

FACTORING IN QUADRATIC FIELDS

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1. INTRODUCTION

For a squarefree integer d other than 1, let

$$K = \mathbf{Q}[\sqrt{d}] = \{x + y\sqrt{d} : x, y \in \mathbf{Q}\}.$$

This is closed under addition, subtraction, multiplication, and also division by nonzero numbers. Similarly, set

$$\mathbf{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbf{Z}\}.$$

This is closed under addition, subtraction, and multiplication, but not usually division.

We will define a concept of “integers” for K , which will play the same role in K as the ordinary integers \mathbf{Z} do in \mathbf{Q} . The integers of K will contain $\mathbf{Z}[\sqrt{d}]$ but may be larger. Unique factorization in the integers of K does not always hold, but we can recover unique factorization if we broaden our view of what we should be trying to factor.

2. CONJUGATION

For $\alpha = x + y\sqrt{d} \in K$ with $x, y \in \mathbf{Q}$, set its *conjugate* to be

$$\bar{\alpha} = x - y\sqrt{d}.$$

Check the following properties by a direct calculation:

$$(2.1) \quad \overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}, \quad \overline{\alpha\beta} = \bar{\alpha} \cdot \bar{\beta}, \quad \overline{\bar{\alpha}} = \alpha,$$

and also

$$(2.2) \quad \bar{\alpha} = \alpha \iff \alpha \in \mathbf{Q}.$$

For example, $\alpha + \bar{\alpha}$ and $\alpha\bar{\alpha}$ are rational for $\alpha \in K$, either because $\alpha + \bar{\alpha}$ and $\alpha\bar{\alpha}$ are fixed by conjugation (use the algebraic properties in (2.1) to check that) or because one can explicitly compute

$$(2.3) \quad \alpha + \bar{\alpha} = 2x, \quad \alpha\bar{\alpha} = x^2 - dy^2$$

where $\alpha = x + y\sqrt{d}$.

Definition 2.1. For $\alpha \in K$, set $\text{Tr}(\alpha) = \alpha + \bar{\alpha}$ and $N(\alpha) = \alpha\bar{\alpha}$. These are called the *trace* and *norm* of α .

Explicit formulas for the trace and norm are in (2.3). Note $\text{Tr}: K \rightarrow \mathbf{Q}$ and $N: K \rightarrow \mathbf{Q}$. For $q \in \mathbf{Q}$, $\text{Tr}(q) = 2q$ and $N(q) = q^2$.

Theorem 2.2. *The trace is additive and the norm is multiplicative. That is, $\text{Tr}(\alpha + \beta) = \text{Tr}(\alpha) + \text{Tr}(\beta)$ and $N(\alpha\beta) = N(\alpha)N(\beta)$.*

Proof. Just compute. □

Every $\alpha \in K$ is the root of a monic polynomial of degree 2 with rational coefficients:

$$(2.4) \quad (X - \alpha)(X - \bar{\alpha}) = X^2 - (\alpha + \bar{\alpha})X + \alpha\bar{\alpha} = X^2 - \text{Tr}(\alpha)X + \text{N}(\alpha).$$

This has α and $\bar{\alpha}$ as its two roots, and the coefficients are the trace (up to sign) and the norm. The coefficients of (2.4) might not be in \mathbf{Z} : if $\alpha = 1/2$ then (2.4) is $X^2 - X + 1/4$.

3. INTEGERS IN A QUADRATIC FIELD

Restricting attention to those elements of K whose polynomial in (2.4) has coefficients in \mathbf{Z} will define for us what the integers of K are.

Definition 3.1. An element $\alpha \in K$ is called an *integer* of K if the polynomial (2.4) has coefficients in \mathbf{Z} . Equivalently, α is an integer of K precisely when its trace and norm are in \mathbf{Z} .

Example 3.2. If α lies in $\mathbf{Z}[\sqrt{d}]$ then (2.4) has integer coefficients, so α is an integer of K .

Example 3.3. If $d \equiv 1 \pmod{4}$ then $\frac{1+\sqrt{d}}{2}$ is not in $\mathbf{Z}[\sqrt{d}]$ but it is an integer of $\mathbf{Q}[\sqrt{d}]$ since it is a root of $X^2 - X + \frac{1-d}{4}$, whose coefficients are in \mathbf{Z} .

Theorem 3.4. *The integers of K are*

$$\{a + b\sqrt{d} : a, b \in \mathbf{Z}\} \text{ if } d \not\equiv 1 \pmod{4}$$

and

$$\left\{ a + b \left(\frac{1 + \sqrt{d}}{2} \right) : a, b \in \mathbf{Z} \right\} \text{ if } d \equiv 1 \pmod{4}.$$

Proof. A direct calculation shows that both of the sets consist of integers in K . (To treat the second case, write the elements as $(a + \frac{b}{2}) + \frac{b}{2}\sqrt{d}$ to calculate the trace and norm using (2.3).) We have to show, conversely, that every integer of K has the indicated form.

Let $\alpha = x + y\sqrt{d}$ be an integer of K , so $2x \in \mathbf{Z}$ and $x^2 - dy^2 \in \mathbf{Z}$. The first condition means x is half an ordinary integer ($x = \frac{1}{2}(2x)$), so either $x \in \mathbf{Z}$ or x is half an odd number.

The importance of d being squarefree is that if $dr^2 \in \mathbf{Z}$ for some $r \in \mathbf{Q}$ then $r \in \mathbf{Z}$. Indeed, if r has a prime factor p in its (reduced form) denominator then $dr^2 \notin \mathbf{Z}$ since p^2 is not canceled by d in dr^2 . So $dr^2 \in \mathbf{Z}$ implies r has no prime in its denominator, *i.e.*, $r \in \mathbf{Z}$.

Suppose x is half an odd number, so $x = a/2$ with a odd. We will show y is also half an odd number and $d \equiv 1 \pmod{4}$. Since $x^2 - dy^2 = a^2/4 - dy^2$ is in \mathbf{Z} , multiplying by 4 implies

$$(3.1) \quad a^2 - d(2y)^2 \in 4\mathbf{Z}$$

Thus $d(2y)^2 \in \mathbf{Z}$, so $2y \in \mathbf{Z}$ since d is squarefree. Either $y \in \mathbf{Z}$ or y is half an odd number. If $y \in \mathbf{Z}$ then $a^2 - d(2y)^2 = a^2 - 4dy^2$ is odd (recall a is odd), which contradicts (3.1). Thus $y = b/2$ for an odd $b \in \mathbf{Z}$, so $x^2 - dy^2 = (a^2 - db^2)/4 \in \mathbf{Z}$, which implies $a^2 \equiv db^2 \pmod{4}$. All odd squares are 1 mod 4, so the congruence implies $d \equiv 1 \pmod{4}$. Returning to α ,

$$\alpha = x + y\sqrt{d} = \frac{a}{2} + \frac{b}{2}\sqrt{d} = \frac{a-b}{2} + b \left(\frac{1 + \sqrt{d}}{2} \right),$$

which has the necessary form because $a - b$ is even.

We are done when $d \equiv 1 \pmod{4}$. When $d \not\equiv 1 \pmod{4}$ we can't have x be half an odd number, so $x \in \mathbf{Z}$. From $x^2 - dy^2 \in \mathbf{Z}$ it follows that $dy^2 \in \mathbf{Z}$, so from d being squarefree we get $y \in \mathbf{Z}$. Therefore α has the necessary form, with $a = x$ and $b = y$. \square

To unify the notation, set

$$\omega = \begin{cases} \sqrt{d}, & \text{if } d \not\equiv 1 \pmod{4}, \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

Then ω is an integer of K , and the set of all integers of K takes the form

$$\mathbf{Z}[\omega] = \{a + b\omega : a, b \in \mathbf{Z}\}$$

in all cases.

We will denote the integers of K as \mathcal{O}_K , so $\mathcal{O}_K = \mathbf{Z}[\omega]$. Table 1 gives some examples.

K	$\mathbf{Q}[i]$	$\mathbf{Q}[\sqrt{2}]$	$\mathbf{Q}[\sqrt{5}]$	$\mathbf{Q}[\sqrt{-5}]$	$\mathbf{Q}[\sqrt{-14}]$
\mathcal{O}_K	$\mathbf{Z}[i]$	$\mathbf{Z}[\sqrt{2}]$	$\mathbf{Z}[(1+\sqrt{5})/2]$	$\mathbf{Z}[\sqrt{-5}]$	$\mathbf{Z}[\sqrt{-14}]$

TABLE 1. Integers in Quadratic Fields

Remember that $\mathbf{Z}[\sqrt{d}] \subset \mathcal{O}_K$, but when $d \equiv 1 \pmod{4}$ the set \mathcal{O}_K is strictly larger than $\mathbf{Z}[\sqrt{d}]$. Easily \mathcal{O}_K is closed under addition and subtraction. It is closed under multiplication too. To verify that, we have

$$(a + b\omega)(a' + b'\omega) = aa' + (ab' + a'b)\omega + bb'\omega^2,$$

and we need to check that ω^2 is a \mathbf{Z} -linear combination of 1 and ω . If $d \not\equiv 1 \pmod{4}$ then $\omega^2 = d = d \cdot 1$. If $d \equiv 1 \pmod{4}$, then $\omega^2 = \omega + (d-1)/4 = \frac{d-1}{4} \cdot 1 + 1 \cdot \omega$ and $\frac{d-1}{4} \in \mathbf{Z}$ since $d \equiv 1 \pmod{4}$.

Theorem 3.5. *For $m \in \mathbf{Z}$ and $\alpha = a + b\omega \in \mathbf{Z}[\omega]$, $m \mid \alpha$ in $\mathbf{Z}[\omega]$ if and only if $m \mid a$ and $m \mid b$ in \mathbf{Z} .*

Proof. If $m \mid a$ and $m \mid b$ in \mathbf{Z} then easily $m \mid \alpha$. Conversely, if $m \mid \alpha$ then $a + b\omega = m(a' + b'\omega)$ for some a' and b' in \mathbf{Z} . Therefore $a = ma'$ and $b = mb'$, so $m \mid a$ and $m \mid b$ in \mathbf{Z} . \square

We will use Theorem 3.5 often without comment in numerical examples, e.g., $5 + \sqrt{-6}$ is not divisible by 3 in $\mathbf{Z}[\sqrt{-6}]$.

Theorem 3.6. *For each quadratic field K , $\mathcal{O}_K \cap \mathbf{Q} = \mathbf{Z}$ and every element of K is a ratio of elements from \mathcal{O}_K .*

Proof. Each element of \mathcal{O}_K is $a + b\omega$ with a and b in \mathbf{Z} . Since $\omega \notin \mathbf{Q}$, if $a + b\omega \in \mathbf{Q}$ then $b = 0$, so $a + b\omega = a \in \mathbf{Z}$.

To show every element of K can be written as the ratio of two elements of \mathcal{O}_K , if $\alpha = x + y\sqrt{d}$ with rational x and y and we write the fractions x and y with a common denominator, say $x = a/c$ and $y = b/c$ for some $c \in \mathbf{Z}$, then

$$(3.2) \quad x + y\sqrt{d} = \frac{a + b\sqrt{d}}{c}.$$

Since $\mathbf{Z}[\sqrt{d}] \subset \mathcal{O}_K$, we're done. \square

Theorem 3.6 says (in part) that the notion of an integer in a quadratic field does not introduce unexpected new integers inside of \mathbf{Q} : the rational numbers that are integers in K are the plain integers \mathbf{Z} .

Theorem 3.7. *If $\alpha \in \mathcal{O}_K$ then $\bar{\alpha} \in \mathcal{O}_K$.*

Proof. Since α and $\bar{\alpha}$ have the same trace and the same norm, we have $\bar{\alpha} \in \mathcal{O}_K$ directly from Definition 3.1. For an alternate proof, show the sets described by Theorem 3.4 are preserved by conjugation. \square

Although both the trace and norm will be important, the norm will play a more dominant role. The reason is that we are going to be interested in multiplicative questions (like factoring), and the norm turns multiplicative relations in \mathcal{O}_K into multiplicative relations in \mathbf{Z} , where we are more comfortable. The next two theorems illustrate this idea.

Let \mathcal{O}_K^\times denote the elements of \mathcal{O}_K that have multiplicative inverses in \mathcal{O}_K . These are called the *units* of \mathcal{O}_K . For example, i is a unit in $\mathbf{Z}[i]$ (with inverse $-i$) and $1 + \sqrt{2}$ is a unit of $\mathbf{Z}[\sqrt{2}]$ (since $(1 + \sqrt{2})(\sqrt{2} - 1) = 1$). The units of \mathcal{O}_K are closed under multiplication.

Theorem 3.8. *For each quadratic field K , $\mathcal{O}_K^\times = \{\alpha \in \mathcal{O}_K : N(\alpha) = \pm 1\}$ and $\mathcal{O}_K^\times \cap \mathbf{Q} = \{\pm 1\}$.*

Proof. Let $\alpha \in \mathcal{O}_K$. If α is a unit, then $\alpha\beta = 1$ for some $\beta \in \mathcal{O}_K$. Taking norms of both sides, $N(\alpha)N(\beta) = N(1) = 1$ in \mathbf{Z} , so $N(\alpha) = \pm 1$. Conversely, assume $N(\alpha) = \pm 1$. Since $N(\alpha) = \alpha\bar{\alpha}$, we get $\alpha\bar{\alpha} = \pm 1$. Therefore $\pm\bar{\alpha}$ is an inverse for α , and this lies in \mathcal{O}_K by Theorem 3.7.

To show $\mathcal{O}_K^\times \cap \mathbf{Q} = \{\pm 1\}$, the inclusion \supset is obvious. For the inclusion \subset , let $q \in \mathcal{O}_K^\times \cap \mathbf{Q}$. Then $N(q) = \pm 1$ since $q \in \mathcal{O}_K^\times$, so $q^2 = \pm 1$ since q is rational. Thus $q = \pm 1$. \square

Example 3.9. The units in $\mathbf{Z}[\sqrt{-14}]$ are ± 1 : for $a, b \in \mathbf{Z}$, $a + b\sqrt{-14}$ has norm $a^2 + 14b^2$, which is never -1 and is 1 only for $a = \pm 1$ and $b = 0$.

We say a nonzero $\alpha \in \mathcal{O}_K$ is *irreducible* if α is not a unit and every factorization $\alpha = \beta\gamma$ in \mathcal{O}_K requires β or γ is a unit in \mathcal{O}_K .

Theorem 3.10. *If $\alpha \in \mathcal{O}_K$ has a norm that is prime in \mathbf{Z} then α is irreducible in \mathcal{O}_K .*

Proof. Suppose $\alpha = \beta\gamma$ with β and γ in \mathcal{O}_K . Then taking norms of both sides gives us $N(\alpha) = N(\beta)N(\gamma)$ in \mathbf{Z} . Since $N(\alpha)$ is prime, either $N(\beta)$ or $N(\gamma)$ is ± 1 , so (by Theorem 3.8) either β or γ is a unit in \mathcal{O}_K . Thus α doesn't have a nontrivial factorization in \mathcal{O}_K , so it is irreducible. \square

Example 3.11. In $\mathbf{Z}[\sqrt{-14}]$, $3 + \sqrt{-14}$ has norm 23 , so $3 + \sqrt{-14}$ is irreducible in $\mathbf{Z}[\sqrt{-14}]$.

Remark 3.12. Negative primes are allowed in Theorem 3.10. For instance, in $\mathbf{Z}[\sqrt{3}]$ the norm of $1 + 2\sqrt{3}$ is -11 , so $1 + 2\sqrt{3}$ is irreducible.

The criterion in Theorem 3.10 is sufficient to show an element of \mathcal{O}_K is irreducible, but it is *not* necessary.

Example 3.13. In $\mathbf{Z}[\sqrt{-14}]$, $N(3) = 9$ is not prime in \mathbf{Z} , but 3 is irreducible in $\mathbf{Z}[\sqrt{-14}]$. Indeed, suppose $3 = \alpha\beta$ in $\mathbf{Z}[\sqrt{-14}]$ with non-units α and β . Taking norms of both sides, $9 = N(\alpha)N(\beta)$ in \mathbf{Z} . The norms of α and β must be 3 (they are positive since norms look like $a^2 + 14b^2$, and they are not 1 since α and β are not units), but the equation $3 = a^2 + 14b^2$ has no solutions in \mathbf{Z} , so there are no elements with norm 3 .

Similarly, since $N(5) = 25$ and 5 is not a norm from $\mathbf{Z}[\sqrt{-14}]$, 5 is irreducible in $\mathbf{Z}[\sqrt{-14}]$.

Example 3.14. The norm of $1 + \sqrt{-14}$ is 15 , which factors in \mathbf{Z} , but $1 + \sqrt{-14}$ is irreducible in $\mathbf{Z}[\sqrt{-14}]$. To see why, write $1 + \sqrt{-14} = \alpha\beta$ and take norms to get $15 = N(\alpha)N(\beta)$ in \mathbf{Z} . Since 3 and 5 are not norms from $\mathbf{Z}[\sqrt{-14}]$, one of α or β has norm 1 , so α or β is a unit.

Theorem 3.15. *Every nonzero non-unit in \mathcal{O}_K is a product of irreducibles in \mathcal{O}_K .*

Proof. We argue by induction on $|N(\alpha)|$, where α runs over nonzero non-units in \mathcal{O}_K . If $|N(\alpha)| = 2$ then α has prime norm so α is irreducible and thus is its own irreducible factorization. Suppose $|N(\alpha)| = n \geq 3$ and all elements with norm of absolute value from 2 to $n-1$ admit a factorization into irreducibles. If α is irreducible then it has an irreducible factorization. If α is not irreducible then we can write $\alpha = \beta\gamma$ where β and γ are not units. Therefore $|N(\beta)|$ and $|N(\gamma)|$ are both less than $|N(\alpha)|$, so by the inductive hypothesis we have

$$\beta = \pi_1 \cdots \pi_r, \quad \gamma = \pi'_1 \cdots \pi'_{r'},$$

where π_i and π'_j are irreducible in \mathcal{O}_K . Thus

$$\alpha = \beta\gamma = \pi_1 \cdots \pi_r \pi'_1 \cdots \pi'_{r'}$$

is a product of irreducibles. □

For some quadratic fields (such as $\mathbf{Q}[i]$ and $\mathbf{Q}[\sqrt{2}]$), their integers have unique factorization into irreducibles. But very often \mathcal{O}_K does not have unique factorization.

Example 3.16. In $\mathbf{Z}[\sqrt{-14}]$, 15 has the two factorizations

$$(3.3) \quad 3 \cdot 5 = (1 + \sqrt{-14})(1 - \sqrt{-14}).$$

These are irreducible factorizations by Examples 3.13 and 3.14. No factor in one product is a unit multiple of a factor in the other product since the units in $\mathbf{Z}[\sqrt{-14}]$ are ± 1 .

The irreducible element 3 divides $(1 + \sqrt{-14})(1 - \sqrt{-14})$, but it does not divide either factor. This is not like the behavior of primes p in \mathbf{Z} , where $p \mid ab$ always implies $p \mid a$ or $p \mid b$.

Example 3.17. Here is a much more striking instance of non-unique factorization in $\mathbf{Z}[\sqrt{-14}]$:

$$(3.4) \quad 3 \cdot 3 \cdot 3 \cdot 3 = (5 + 2\sqrt{-14})(5 - 2\sqrt{-14}).$$

What makes (3.4) more interesting than (3.3) is that the number of irreducible factors on both sides is not the same. To see that $5 + 2\sqrt{-14}$ is irreducible in $\mathbf{Z}[\sqrt{-14}]$, if it has a non-unit proper factor then that factor has norm properly dividing $N(5 + 2\sqrt{-14}) = 81$, so the norm is 3, 9, or 27. No element has norm 3 or 27, and the elements of norm 9 are ± 3 , neither of which are factors of $5 + 2\sqrt{-14}$. The same proof shows $5 - 2\sqrt{-14}$ is irreducible.

Example 3.18. In \mathbf{Z} , when relatively prime numbers have a product that is a perfect square, the two numbers are squares up to multiplication by ± 1 . This is proved using unique factorization in \mathbf{Z} . In $\mathbf{Z}[\sqrt{-14}]$, where there is no unique factorization, the corresponding result is false. Let's take a look at a simple example.

Consider the equation

$$2 \cdot (-7) = \sqrt{-14}^2.$$

The factors on the left have no common factor in $\mathbf{Z}[\sqrt{-14}]$ besides ± 1 (if $\delta \mid 2$ and $\delta \mid (-7)$ then δ divides $-7(-1) - 3 \cdot 2 = 1$, so δ is a unit), and their product is a perfect square, but neither factor is a square up to a unit multiple: if $2 = \pm(a + b\sqrt{-14})^2 = \pm(a^2 - 14b^2 + 2ab\sqrt{-14})$ then $ab = 0$ so a or b is 0, but then $2 \neq \pm(a^2 - 14b^2)$. Similarly, 7 is not a square up to unit multiple.

4. IDEALS

Instead of working with elements in \mathcal{O}_K , where unique factorization can fail, we will develop a multiplicative theory for certain subsets of \mathcal{O}_K , called (for historical reasons) ideals.

Definition 4.1. For a finite set of elements $\alpha_1, \dots, \alpha_m$ in \mathcal{O}_K , the set

$$\begin{aligned} (\alpha_1, \dots, \alpha_m) &:= \{\alpha_1\gamma_1 + \dots + \alpha_m\gamma_m : \gamma_i \in \mathcal{O}_K\} \\ &= \alpha_1\mathcal{O}_K + \dots + \alpha_m\mathcal{O}_K \end{aligned}$$

is called an *ideal* in \mathcal{O}_K and we call $\alpha_1, \dots, \alpha_m$ *generators* of that ideal. The ideals of the form $(\alpha) = \alpha\mathcal{O}_K$, with a single generator, are called *principal ideals*.

Example 4.2. In $\mathbf{Z}[\sqrt{10}]$, $(6, \sqrt{10})$ is the set of all linear combinations $6\gamma_1 + \sqrt{10}\gamma_2$ where γ_1 and γ_2 run over $\mathbf{Z}[\sqrt{10}]$. Taking $\gamma_1 = 1$ and $\gamma_2 = 0$ shows 6 is in the ideal. Similarly, $\sqrt{10}$ is in this ideal.

An ideal in \mathcal{O}_K is the set of all \mathcal{O}_K -linear combinations of a finite set of elements in \mathcal{O}_K . In particular, an ideal $(\alpha_1, \dots, \alpha_m)$ contains each α_i . Observe that $\beta \in (\alpha)$ if and only if $\alpha \mid \beta$.

The order in which we write down the generators of an ideal doesn't matter, *e.g.*, $(\alpha_1, \alpha_2, \alpha_3) = (\alpha_3, \alpha_1, \alpha_2)$. What is much more important to remember, though, is that different finite sets can produce the same ideal.

Example 4.3. In $\mathbf{Z}[\sqrt{-14}]$, we will show

$$(17 + 2\sqrt{-14}, 20 + \sqrt{-14}) = (3 - \sqrt{-14}).$$

The right side is all $\mathbf{Z}[\sqrt{-14}]$ -multiples of $3 - \sqrt{-14}$. The left side is the $\mathbf{Z}[\sqrt{-14}]$ -linear combinations of $17 + 2\sqrt{-14}$ and $20 + \sqrt{-14}$. Since $17 + 2\sqrt{-14} = (3 - \sqrt{-14})(1 + \sqrt{-14})$ and $20 + \sqrt{-14} = (3 - \sqrt{-14})(2 + \sqrt{-14})$, $3 - \sqrt{-14}$ is a common divisor of the two generators on the left and therefore is a factor of each $\mathbf{Z}[\sqrt{-14}]$ -linear combination of these numbers, which means every element of the ideal on the left is in the ideal on the right. Conversely, since

$$(3 - \sqrt{-14})\gamma = (20 + \sqrt{-14})\gamma + (17 + 2\sqrt{-14})(-\gamma),$$

each element in the ideal on the right is also in the ideal on the left. Thus these two ideals are the same.

Example 4.4. We show the ideal $(2, \sqrt{-14})$ in $\mathbf{Z}[\sqrt{-14}]$ is not principal. The first thing to notice is that every element of $(2, \sqrt{-14})$ has an even norm:

$$2(a + b\sqrt{-14}) + \sqrt{-14}(a' + b'\sqrt{-14}) = (2a - 14b') + (2b + a')\sqrt{-14}$$

has norm

$$2^2(a - 7b')^2 + 14(2b + a')^2,$$

which is even.

Suppose $(2, \sqrt{-14}) = (\alpha)$. Then, since $2 \in (2, \sqrt{-14})$ we have $2 \in (\alpha)$, so $\alpha \mid 2$ in $\mathbf{Z}[\sqrt{-14}]$. Writing $2 = \alpha\beta$ in $\mathbf{Z}[\sqrt{-14}]$ and taking norms, $4 = N(\alpha)N(\beta)$ in \mathbf{Z} , so $N(\alpha) \mid 4$ in \mathbf{Z} . Similarly, since $\sqrt{-14} \in (\alpha)$ we get $N(\alpha) \mid 14$ in \mathbf{Z} . Thus $N(\alpha)$ is a common divisor of 4 and 14, so $N(\alpha)$ is 1 or 2. The norm's values are $x^2 + 14y^2$ with $x, y \in \mathbf{Z}$, which is never 2. Therefore $N(\alpha) = 1$, so α is a unit and $(\alpha) = (1)$. But that means $1 \in (2, \sqrt{-14})$, contradicting the fact that every element of $(2, \sqrt{-14})$ has even norm. Hence $(2, \sqrt{-14})$ is not principal.

We will generally denote ideals with small gothic letters like \mathfrak{a} and \mathfrak{b} . The key algebraic properties of an ideal are twofold. First, an ideal is closed under addition. Second, an ideal “swallows up” multiplication by each element of \mathcal{O}_K . That is, if \mathfrak{a} is an ideal and $\alpha \in \mathfrak{a}$ then $\gamma\alpha \in \mathfrak{a}$ for all $\gamma \in \mathcal{O}_K$. The reason is that an \mathcal{O}_K -multiple of an \mathcal{O}_K -linear combination of numbers is again an \mathcal{O}_K -linear combination of the same numbers. *Make sure you understand that.*

Example 4.5. The whole set \mathcal{O}_K is the ideal (1). More generally, an ideal containing a unit has to be the ideal (1). After all, an ideal contains every \mathcal{O}_K -multiple of every element inside it, and all of \mathcal{O}_K is a multiple of each unit. We call (1) the *unit ideal*.

Since ideals can be described with different sets of generators, we will avoid defining concepts about ideals in terms of a choice of generators. However, since it is the generator description that we always make computations with, we will always try to check how a new definition looks in terms of generators for the ideals involved.

Theorem 4.6. *Let $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and $\mathfrak{b} = (\beta_1, \dots, \beta_n)$ be two ideals in \mathcal{O}_K . Then the following are equivalent:*

- (a) $\mathfrak{a} \subset \mathfrak{b}$,
- (b) *each α_i is in \mathfrak{b} ,*
- (c) *each α_i is an \mathcal{O}_K -linear combination of the β_j 's.*

Proof. If $\mathfrak{a} \subset \mathfrak{b}$ then each α_i belongs to \mathfrak{b} , which means each α_i is an \mathcal{O}_K -linear combination of the β_j 's. This shows (a) \Rightarrow (b) \Rightarrow (c). Finally, if each α_i is an \mathcal{O}_K -linear combination of the β_j 's then each α_i is in \mathfrak{b} , so (since \mathfrak{b} is closed under \mathcal{O}_K -scaling and addition) all sums of \mathcal{O}_K -multiples of the different α_i 's are in \mathfrak{b} . That is what a typical element of \mathfrak{a} looks like, so $\mathfrak{a} \subset \mathfrak{b}$. \square

Corollary 4.7. *In the notation of Theorem 4.6, we have $\mathfrak{a} = \mathfrak{b}$ if and only if every α_i is an \mathcal{O}_K -linear combination of the β_j 's and every β_j is an \mathcal{O}_K -linear combination of the α_i 's.*

Proof. To say $\mathfrak{a} = \mathfrak{b}$ means $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a}$. Use Theorem 4.6 to interpret these inclusions in terms of linear combinations. \square

Example 4.8. For α_1 and α_2 in \mathcal{O}_K , $(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2 + \gamma\alpha_1)$ for all $\gamma \in \mathcal{O}_K$.

Example 4.9. In $\mathbf{Z}[\sqrt{-14}]$, $(2, 1 + \sqrt{-14}) = (1)$ because $(2, 1 + \sqrt{-14})$ contains $(1 + \sqrt{-14})(1 - \sqrt{-14}) = 15$, which is relatively prime to 2 in \mathbf{Z} , so we know how to write 1 as a combination of 2 and 15:

$$1 = 15 - 7 \cdot 2 = (1 - \sqrt{-14})(1 + \sqrt{-14}) - 7 \cdot 2.$$

This exhibits 1 as an element of $(2, 1 + \sqrt{-14})$, so this is the unit ideal.

Example 4.10. In $\mathbf{Z}[\sqrt{-14}]$,

$$(2 + \sqrt{-14}, 7 + 2\sqrt{-14}) = (3, 1 - \sqrt{-14}).$$

because

$$(7 + 2\sqrt{-14}) - 2(2 + \sqrt{-14}) = 3, \quad (7 + 2\sqrt{-14}) - 3(2 + \sqrt{-14}) = 1 - \sqrt{-14}$$

and

$$3 - (1 - \sqrt{-14}) = 2 + \sqrt{-14}, \quad 3 \cdot 3 - 2(1 - \sqrt{-14}) = 7 + 2\sqrt{-14}.$$

Is the ideal $(3, 1 - \sqrt{-14})$ the unit ideal? No, because as in Example 4.4 a calculation shows every element of $(3, 1 - \sqrt{-14})$ has norm divisible by 3, so $1 \notin (3, 1 - \sqrt{-14})$.

Example 4.11. In $\mathbf{Z}[\sqrt{-14}]$,

$$(4 + \sqrt{-14}, 2 - \sqrt{-14}, 7 - 2\sqrt{-14}, 7 + \sqrt{-14}) = (3, 1 + \sqrt{-14}).$$

To show the right side is in the left side, $-2(2 - \sqrt{-14}) + (7 - 2\sqrt{-14}) = 3$ and $2(4 + \sqrt{-14}) - (7 + \sqrt{-14}) = 1 + \sqrt{-14}$. It is left to the reader to express each generator on the left side as a $\mathbf{Z}[\sqrt{-14}]$ -linear combination of 3 and $1 + \sqrt{-14}$.

Theorem 4.12. *If an ideal in \mathcal{O}_K contains two elements of \mathbf{Z} that are relatively prime then the ideal is the unit ideal. In particular, an ideal is the unit ideal if it contains two elements whose norms are relatively prime in \mathbf{Z} .*

Proof. Let \mathfrak{a} be an ideal and a and b be elements of \mathfrak{a} that are in \mathbf{Z} and relatively prime. We can write $1 = ax + by$ for some x and y in \mathbf{Z} . The right side is in \mathfrak{a} , so $1 \in \mathfrak{a}$, so $\mathfrak{a} = (1)$.

Since the norm of each $\alpha \in \mathfrak{a}$ is also in \mathfrak{a} (because $N(\alpha)$ is a multiple of α and ideals contain each multiple of their elements), two relatively prime norms of elements in \mathfrak{a} are themselves elements of \mathfrak{a} . So $\mathfrak{a} = (1)$ by the previous paragraph. \square

Theorem 4.13. *An ideal in \mathcal{O}_K that has a set of generators from \mathbf{Z} is a principal ideal.*

Proof. Let $\mathfrak{a} = (a_1, \dots, a_m)$ where $a_i \in \mathbf{Z}$. Let a be the greatest common divisor of the a_i 's in \mathbf{Z} . Then every a_i is a \mathbf{Z} -multiple of a , so each element of \mathfrak{a} is divisible by a in \mathcal{O}_K . This shows $\mathfrak{a} \subset (a) = a\mathcal{O}_K$. Conversely, generalizing the fact that the greatest common divisor of two integers is a \mathbf{Z} -linear combination of them, it is possible to write $a = c_1a_1 + \dots + c_ma_m$ for some $c_i \in \mathbf{Z}$ (check!), so each \mathcal{O}_K -multiple of a is an \mathcal{O}_K -linear combination of the a_i 's. This shows $(a) \subset \mathfrak{a}$, so $\mathfrak{a} = (a)$. \square

The point of Theorem 4.13 is that it tells us that if we happen to find an ideal with generators all taken from \mathbf{Z} , we can write the ideal in a much simpler form with a single generator from \mathbf{Z} . This will play a role in the key theorem about ideals (Theorem 5.4).

Theorem 4.14. *For α and β in \mathcal{O}_K , $(\alpha) = (\beta)$ if and only if α and β are equal up to multiplication by a unit in \mathcal{O}_K .*

Proof. Since $(\alpha) = (0)$ if and only if $\alpha = 0$ (a principal ideal with a nonzero generator contains a nonzero element), we may assume that the common principal ideal $(\alpha) = (\beta)$ is not (0) since otherwise we are in a trivial case.

By Corollary 4.7, $(\alpha) = (\beta)$ is equivalent to $\alpha \in (\beta)$ and $\beta \in (\alpha)$, i.e., α being a multiple of β and β being a multiple of α . Writing $\alpha = \beta\gamma$ and $\beta = \alpha\gamma'$, this requires $\alpha = \alpha\gamma'\gamma$. Since $\alpha \neq 0$ we get $1 = \gamma'\gamma$, so γ and γ' are units in \mathcal{O}_K . Thus α and β are equal up to unit multiple. Conversely, if α and β are equal up to unit multiple then they are multiples of each other so $(\alpha) = (\beta)$. \square

Now we define multiplication of ideals.

Definition 4.15. For ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O}_K , the *product* $\mathfrak{a}\mathfrak{b}$ is the set of all finite sums $\sum_{k=1}^r x_k y_k$, with $r \geq 1$, $x_k \in \mathfrak{a}$ and $y_k \in \mathfrak{b}$.

Where does this definition come from and why is $\mathfrak{a}\mathfrak{b}$ an ideal? Well, whatever the product of \mathfrak{a} and \mathfrak{b} ought to mean, it should at least be an ideal containing the pairwise products xy where $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. Then, since an ideal has to be closed under addition, the product of \mathfrak{a} and \mathfrak{b} should include all finite sums $\sum_{k=1}^r x_k y_k$ with $x_k \in \mathfrak{a}$ and $y_k \in \mathfrak{b}$. This is exactly what Definition 4.15 is about. The following theorem shows, in terms of generators, why $\mathfrak{a}\mathfrak{b}$ is an ideal.

Theorem 4.16. *If $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and $\mathfrak{b} = (\beta_1, \dots, \beta_n)$ then*

$$\mathfrak{ab} = (\alpha_1\beta_1, \dots, \alpha_i\beta_j, \dots, \alpha_m\beta_n).$$

Proof. Every element of \mathfrak{ab} has the form $x_1y_1 + \dots + x_ry_r$ where $x_k \in \mathfrak{a}$ and $y_k \in \mathfrak{b}$. We can write each x_k as an \mathcal{O}_K -linear combination of the α_i 's and each y_k as an \mathcal{O}_K -linear combination of the β_j 's. Multiplying out the product x_ky_k shows it is an \mathcal{O}_K -linear combination of the $\alpha_i\beta_j$'s. Then a sum of such products is another \mathcal{O}_K -linear combination of the $\alpha_i\beta_j$'s, so the elements of \mathfrak{ab} do lie in the ideal $(\alpha_1\beta_1, \dots, \alpha_i\beta_j, \dots, \alpha_m\beta_n)$. Conversely, every element of this ideal is an \mathcal{O}_K -linear combination of the $\alpha_i\beta_j$'s, so it is a sum

$$(4.1) \quad \sum_{i=1}^m \sum_{j=1}^n \gamma_{ij} \alpha_i \beta_j$$

where $\gamma_{ij} \in \mathcal{O}_K$. Since $\gamma_{ij}\alpha_i \in \mathfrak{a}$ and $\beta_j \in \mathfrak{b}$, the sum (4.1) is of the form $\sum_{k=1}^r x_k y_k$ with $x_k \in \mathfrak{a}$ and $y_k \in \mathfrak{b}$, so the sum belongs to \mathfrak{ab} . \square

Example 4.17. We compute a product of ideals in $\mathbf{Z}[\sqrt{-14}]$. Let $\mathfrak{a} = (5 + \sqrt{-14}, 2 + \sqrt{-14})$ and $\mathfrak{b} = (4 + \sqrt{-14}, 2 - \sqrt{-14})$. Then

$$\begin{aligned} \mathfrak{ab} &= (5 + \sqrt{-14}, 2 + \sqrt{-14})(4 + \sqrt{-14}, 2 - \sqrt{-14}) \\ &= (6 + 9\sqrt{-14}, -6 + 6\sqrt{-14}, 24 - 3\sqrt{-14}, 18). \end{aligned}$$

This can be simplified to $(6, 3\sqrt{-14})$. Each of the four generators above is easily a linear combination of 6 and $3\sqrt{-14}$ (even with \mathbf{Z} -coefficients, but we always allow for $\mathbf{Z}[\sqrt{-14}]$ -coefficients). For the reverse inclusion, from the first two generators we can get (by doubling the first, tripling the second, and subtracting) 30 in the ideal. Since 18 is in the ideal, as the last generator, so is $6 = 18 \cdot 2 - 30$. Once we have 6 in the ideal we immediately get $3\sqrt{-14} = 6 \cdot 4 - (24 - 3\sqrt{-14})$ in the ideal, so $\mathfrak{ab} = (6, 3\sqrt{-14})$.

Corollary 4.18. *For ideals \mathfrak{a} and \mathfrak{b} , $\mathfrak{ab} = (0)$ if and only if $\mathfrak{a} = (0)$ or $\mathfrak{b} = (0)$.*

Proof. If $\mathfrak{a} = (0)$ or $\mathfrak{b} = (0)$ then the product \mathfrak{ab} is (0) from the formula in Theorem 4.16. If $\mathfrak{a} \neq (0)$ and $\mathfrak{b} \neq (0)$, then \mathfrak{a} has a nonzero element x and \mathfrak{b} has a nonzero element y . Then \mathfrak{ab} contains xy , which is not zero, so $\mathfrak{ab} \neq (0)$. \square

Corollary 4.19. *Multiplication of ideals is commutative and associative. That is, for ideals \mathfrak{a} , \mathfrak{b} , and \mathfrak{c} in \mathcal{O}_K ,*

$$\mathfrak{ab} = \mathfrak{ba}, \quad (\mathfrak{ab})\mathfrak{c} = \mathfrak{a}(\mathfrak{bc}).$$

Proof. Let \mathfrak{a} have generators α_i , \mathfrak{b} have generators β_j , and \mathfrak{c} have generators γ_k . Then, by Theorem 4.16, \mathfrak{ab} is the ideal with generators $\alpha_i\beta_j$ and \mathfrak{ba} is the ideal with generators $\beta_j\alpha_i$. These are the same sets of generators, so $\mathfrak{ab} = \mathfrak{ba}$. Generators of $(\mathfrak{ab})\mathfrak{c}$ are the products $(\alpha_i\beta_j)\gamma_k$ and generators for $\mathfrak{a}(\mathfrak{bc})$ are the products $\alpha_i(\beta_j\gamma_k)$. These are the same generators, so $(\mathfrak{ab})\mathfrak{c} = \mathfrak{a}(\mathfrak{bc})$. \square

Corollary 4.20. *For an ideal $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and a principal ideal (γ) ,*

$$(\gamma)\mathfrak{a} = (\gamma\alpha_1, \dots, \gamma\alpha_m).$$

In particular, the unit ideal $(1) = \mathcal{O}_K$ is an identity for multiplication and $(\alpha)(\beta) = (\alpha\beta)$.

Proof. This is immediate from Theorem 4.16. \square

Example 4.21. In Example 4.4 we showed the ideal $(2, \sqrt{-14})$ in $\mathbf{Z}[\sqrt{-14}]$ is not principal. We will show this in a different way now. Squaring the ideal,

$$(2, \sqrt{-14})^2 = (2, \sqrt{-14})(2, \sqrt{-14}) = (4, 2\sqrt{-14}, -14) = (2)(2, \sqrt{-14}, -7).$$

Since 2 and 7 are relatively prime in \mathbf{Z} , $(2, \sqrt{-14}, -7) = (1)$ by Theorem 4.12. Therefore

$$(4.2) \quad (2, \sqrt{-14})^2 = (2)(1) = (2).$$

If $(2, \sqrt{-14}) = (\alpha)$ then $(2) = (\alpha)^2 = (\alpha^2)$, so $\alpha^2 = \pm 2$. Taking norms, $N(\alpha)^2 = 4$, so $N(\alpha) = 2$. But no element of $\mathbf{Z}[\sqrt{-14}]$ has norm 2, so we have a contradiction.

Definition 4.22. For an ideal \mathfrak{a} , its conjugate ideal is $\bar{\mathfrak{a}} := \{\bar{\alpha} : \alpha \in \mathfrak{a}\}$.

The next theorem gives a formula for $\bar{\mathfrak{a}}$ in terms of generators of \mathfrak{a} , thereby showing that $\bar{\mathfrak{a}}$ is an ideal.

Theorem 4.23. If $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ then $\bar{\mathfrak{a}} = (\bar{\alpha}_1, \dots, \bar{\alpha}_m)$. In particular, if $\mathfrak{a} = (\alpha)$ is principal then $\bar{\mathfrak{a}} = (\bar{\alpha})$ is also principal. For all ideals \mathfrak{a} and \mathfrak{b} , $\overline{\mathfrak{a}\mathfrak{b}} = \bar{\mathfrak{a}}\bar{\mathfrak{b}}$ and $\overline{\bar{\mathfrak{a}}} = \mathfrak{a}$.

Proof. Each element of \mathfrak{a} has the form $\sum_{i=1}^m \gamma_i \alpha_i$ with $\gamma_i \in \mathcal{O}_K$. Its conjugate is $\sum_{i=1}^m \bar{\gamma}_i \bar{\alpha}_i$, so $\bar{\mathfrak{a}} \subset (\bar{\alpha}_1, \dots, \bar{\alpha}_m)$. Every element of $(\bar{\alpha}_1, \dots, \bar{\alpha}_m)$ is a sum $\sum_{i=1}^m \delta_i \bar{\alpha}_i$, which is the conjugate of $\sum_{i=1}^m \bar{\delta}_i \alpha_i \in \mathfrak{a}$, so $(\bar{\alpha}_1, \dots, \bar{\alpha}_m) \subset \bar{\mathfrak{a}}$. Thus $\bar{\mathfrak{a}} = (\bar{\alpha}_1, \dots, \bar{\alpha}_m)$.

The rest of the theorem is left to the reader to prove. \square

Example 4.24. When an *element* of \mathcal{O}_K is equal to its conjugate then it is in \mathbf{Z} . But when an *ideal* in \mathcal{O}_K is equal to its conjugate, it need not be an ideal with generators from \mathbf{Z} . For instance, in $\mathbf{Z}[\sqrt{-14}]$ we have $(2, \sqrt{-14}) = (2, -\sqrt{-14}) = (2, \sqrt{-14})$, so the ideal $(2, \sqrt{-14})$ is equal to its conjugate. This ideal does not have a set of generators from \mathbf{Z} , since if it did then it would be a principal ideal (Theorem 4.13) and we know this ideal is not principal (Example 4.4).

Example 4.25. We will prove that the ideal $(3, 1 + \sqrt{-14})$ in $\mathbf{Z}[\sqrt{-14}]$ is not principal and not equal to its conjugate ideal. To begin, we check that

$$(4.3) \quad (3, 1 + \sqrt{-14})(3, 1 - \sqrt{-14}) = (3).$$

Multiplying together the generators,

$$\begin{aligned} (3, 1 + \sqrt{-14})(3, 1 - \sqrt{-14}) &= (9, 3 - 3\sqrt{-14}, 3 + 3\sqrt{-14}, 15) \\ &= (3)(3, 1 - \sqrt{-14}, 1 + \sqrt{-14}, 5), \end{aligned}$$

and the second ideal on the right contains 3 and 5, which are relatively prime in \mathbf{Z} , so the second ideal is (1). Thus $(3, 1 + \sqrt{-14})(3, 1 - \sqrt{-14}) = (3)(1) = (3)$ and (4.3) is established.

Suppose, to argue by contradiction, that $(3, 1 + \sqrt{-14}) = (\alpha)$ is a principal ideal. Then (4.3) becomes $(\alpha)(\bar{\alpha}) = (3)$. The product $(\alpha)(\bar{\alpha})$ is $(\alpha\bar{\alpha}) = (N(\alpha))$, so for this to be (3) requires $N(\alpha) = \pm 3$. But norms on $\mathbf{Z}[\sqrt{-14}]$ are positive and never equal 3. (The equation $x^2 + 14y^2 = 3$ has no solutions in \mathbf{Z} .) Hence we have a contradiction and $(3, 1 + \sqrt{-14})$ is not principal.

To show $(3, 1 + \sqrt{-14})$ does not equal its conjugate ideal, assume otherwise. Then (4.3) becomes $(3, 1 + \sqrt{-14})^2 = (3)$. But we can compute the square of that ideal independently:

$$\begin{aligned} (3, 1 + \sqrt{-14})^2 &= (3, 1 + \sqrt{-14})(3, 1 + \sqrt{-14}) \\ &= (9, 3 + 3\sqrt{-14}, -13 + 2\sqrt{-14}). \end{aligned}$$

This is not (3) since $-13 + 2\sqrt{-14} \notin (3)$: multiples of 3 in $\mathbf{Z}[\sqrt{-14}]$ have the coefficients of 1 and $\sqrt{-14}$ both divisible by 3.

Having spent some time with multiplication of ideals, we turn to divisibility of ideals.

Definition 4.26. Set $\mathfrak{a} \mid \mathfrak{b}$ if $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ for some ideal \mathfrak{c} . We say \mathfrak{a} *divides* \mathfrak{b} and that \mathfrak{b} is a multiple of \mathfrak{a} , or that \mathfrak{a} is a factor or divisor of \mathfrak{b} .

The first important property of ideal divisibility is that, on principal ideals, it exactly reflects divisibility of the generators as elements of \mathcal{O}_K .

Theorem 4.27. For α and β in \mathcal{O}_K , $(\alpha) \mid (\beta)$ as ideals if and only if $\alpha \mid \beta$ in \mathcal{O}_K .

Proof. Suppose $\alpha \mid \beta$ in \mathcal{O}_K . Then $\beta = \alpha\gamma$ for some $\gamma \in \mathcal{O}_K$. Passing to principal ideals, $(\beta) = (\alpha\gamma) = (\alpha)(\gamma)$, so $(\alpha) \mid (\beta)$. Assume, conversely, that $(\alpha) \mid (\beta)$, so $(\beta) = (\alpha)\mathfrak{c}$ for some ideal \mathfrak{c} . Write $\mathfrak{c} = (\gamma_1, \dots, \gamma_r)$, so

$$(\beta) = (\alpha\gamma_1, \dots, \alpha\gamma_r).$$

Then β , as an element of (β) , is an \mathcal{O}_K -linear combination of the products $\alpha\gamma_k$:

$$\beta = \sum_{k=1}^r \delta_k \alpha \gamma_k = \alpha \sum_{k=1}^r \delta_k \gamma_k,$$

so $\alpha \mid \beta$ in \mathcal{O}_K . □

Theorem 4.28. For $\alpha \in \mathcal{O}_K$ and ideal $\mathfrak{b} = (\beta_1, \dots, \beta_m)$ in \mathcal{O}_K , the following are equivalent:

- $(\alpha) \mid \mathfrak{b}$,
- $\alpha \mid \beta_j$ for all j ,
- $(\alpha) \supset \mathfrak{b}$.

Proof. If $(\alpha) \mid \mathfrak{b}$ then $\mathfrak{b} = (\alpha)\mathfrak{c}$ for some ideal \mathfrak{c} . Write $\mathfrak{c} = (\gamma_1, \dots, \gamma_n)$, so $\mathfrak{b} = (\alpha\gamma_1, \dots, \alpha\gamma_n)$. It follows that every element of \mathfrak{b} is divisible by α , so in particular $\alpha \mid \beta_j$ for each j . Therefore $\beta_j \in (\alpha)$ for all j , so $\mathfrak{b} \subset (\alpha)$. Finally, from this inclusion we can write each β_j as a multiple of α , so \mathfrak{b} has the ideal (α) as a factor. □

Theorem 4.29. For ideals \mathfrak{a} and \mathfrak{b} , if $\mathfrak{a} \mid \mathfrak{b}$ then $\mathfrak{a} \supset \mathfrak{b}$. In particular, if $\mathfrak{a} \mid \mathfrak{b}$ and $\mathfrak{b} \mid \mathfrak{a}$ then $\mathfrak{a} = \mathfrak{b}$.

Proof. Suppose $\mathfrak{a} \mid \mathfrak{b}$, say $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$. Write $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and $\mathfrak{c} = (\gamma_1, \dots, \gamma_n)$. Then \mathfrak{b} is the ideal generated by the products $\alpha_i\gamma_j$. Since ideals, like \mathfrak{a} , swallow up multiplication by all of \mathcal{O}_K , $\alpha_i\gamma_k \in \mathfrak{a}$ for all i and k . Then every sum of \mathcal{O}_K -multiples of these products is in \mathfrak{a} , which means all elements of \mathfrak{b} are in \mathfrak{a} . □

Let's summarize the situation right now. We have replaced multiplication and divisibility among elements of \mathcal{O}_K with multiplication and divisibility among ideals in \mathcal{O}_K . Insofar as elements of \mathcal{O}_K are concerned, their multiplicative and divisibility relations are accurately reflected in the behavior of the principal ideals they generate. This is what the end of Corollary 4.20 and Theorem 4.27 tell us. (Moreover, by Theorem 4.14, replacing elements with principal ideals lets us suppress unit multiple ambiguities.) What do we gain by using ideals in place of elements? We can save unique factorization!

Example 4.30. Let $\mathfrak{p} = (3, 1 + \sqrt{-14})$ and $\mathfrak{q} = (5, 1 + \sqrt{-14})$. We saw in (4.3) that $(3) = \mathfrak{p}\bar{\mathfrak{p}}$. In a similar way, $(5) = \mathfrak{q}\bar{\mathfrak{q}}$, $\mathfrak{p}\mathfrak{q} = (1 + \sqrt{-14})$, and $\bar{\mathfrak{p}}\bar{\mathfrak{q}} = (1 - \sqrt{-14})$. Then the principal ideal (15) can be factored as

$$(15) = (3)(5) = \mathfrak{p}\bar{\mathfrak{p}}\mathfrak{q}\bar{\mathfrak{q}}$$

and as

$$(15) = (1 + \sqrt{-14})(1 - \sqrt{-14}) = \mathfrak{p}\mathfrak{q}\bar{\mathfrak{p}}\bar{\mathfrak{q}}.$$

From the viewpoint of factoring the element 15, (3.3) shows it has non-unique irreducible factorizations. But if we replace those irreducible factors by the principal ideals they generate, they no longer look irreducible (each of the principal ideals (3), (5), $(1 + \sqrt{-14})$, and $(1 - \sqrt{-14})$ factors into a product of ideals as described just above with \mathfrak{p} , \mathfrak{q} , and their conjugate ideals) and in fact the non-unique factorization disappears: all that is happening on the level of ideals is that certain (non-principal) ideals are being multiplied in different ways. Similarly, (3.4) is not strange on the level of ideal factorizations, since $(3)^4 = \mathfrak{p}^4\bar{\mathfrak{p}}^4$, $(5 + 2\sqrt{-14}) = \mathfrak{p}^4$, and $(5 - 2\sqrt{-14}) = \bar{\mathfrak{p}}^4$ for \mathfrak{p} as above. (It is left to the reader to check these formulas for \mathfrak{p}^4 and $\bar{\mathfrak{p}}^4$. As a first step, check $\mathfrak{p}^2 = (9, 2 - \sqrt{-14})$.)

Example 4.31. In $\mathbf{Z}[\sqrt{-14}]$, $2 \cdot (-7) = \sqrt{-14}^2$ is a perfect square and 2 and -7 have no common factors, but 2 and -7 are not perfect squares up to unit multiple (Example 3.18). This mysterious state of affairs is explained when we pass to ideals. Let $\mathfrak{a} = (2, \sqrt{-14})$ and $\mathfrak{b} = (7, \sqrt{-14})$. Then $(2) = \mathfrak{a}^2$ (see (4.2)), $(-7) = (7) = \mathfrak{b}^2$, and

$$\mathfrak{a}\mathfrak{b} = (14, 2\sqrt{-14}, 7\sqrt{-14}, -14) = (\sqrt{-14})(\sqrt{-14}, 2, 7, \sqrt{-14}) = (\sqrt{-14}),$$

so passing from elements to the principal ideals they generate turns the equation $2 \cdot (-7) = \sqrt{-14}^2$ into $\mathfrak{a}^2\mathfrak{b}^2 = (\mathfrak{a}\mathfrak{b})^2$, and now what we expect should be squares are squares, as ideals.

Our main goal is to show that nonzero ideals in \mathcal{O}_K admit unique factorization into what we will call prime ideals. The first step, in the next section, is to establish an analogue for ideals of the cancellation law for non-zero integers.

5. CANCELLING IDEALS

Definition 5.1. An ideal \mathfrak{c} in \mathcal{O}_K is called *cancelable* if whenever $\mathfrak{a}\mathfrak{c} = \mathfrak{b}\mathfrak{c}$ for ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O}_K we have $\mathfrak{a} = \mathfrak{b}$.

Obviously the zero ideal (0) is not cancelable, for the same reason the element 0 can't be cancelled.

Theorem 5.2. *Nonzero principal ideals are cancelable. That is, for nonzero γ in \mathcal{O}_K and ideals \mathfrak{a} and \mathfrak{b} , if $\mathfrak{a}(\gamma) = \mathfrak{b}(\gamma)$ then $\mathfrak{a} = \mathfrak{b}$.*

Proof. We will show $\mathfrak{a} \subset \mathfrak{b}$. The reverse inclusion is handled similarly.

Set $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and $\mathfrak{b} = (\beta_1, \dots, \beta_n)$. Then $\mathfrak{a}(\gamma)$ is generated by the $\alpha_i\gamma$'s and $\mathfrak{b}(\gamma)$ is generated by the $\beta_j\gamma$'s. Therefore each $\alpha_i\gamma$ is an \mathcal{O}_K -linear combination of the $\beta_j\gamma$'s. We can cancel γ to express each α_i as an \mathcal{O}_K -linear combination of the β_j 's, so $\mathfrak{a} \subset \mathfrak{b}$. \square

Corollary 5.3. *Each nonzero ideal in \mathcal{O}_K with a nonzero principal multiple is cancelable.*

Proof. Let \mathfrak{c} be an ideal with a nonzero principal multiple, say $\mathfrak{c}\mathfrak{c}' = (\gamma)$ with $\gamma \neq 0$. Then if $\mathfrak{a}\mathfrak{c} = \mathfrak{b}\mathfrak{c}$, multiply both sides by \mathfrak{c}' to get $\mathfrak{a}(\gamma) = \mathfrak{b}(\gamma)$, so $\mathfrak{a} = \mathfrak{b}$ by Theorem 5.2. \square

It turns out that *every* nonzero ideal in \mathcal{O}_K has a nonzero principal multiple, so by Corollary 5.3 every nonzero ideal in \mathcal{O}_K is cancelable. We're going to show ideals have principal multiples using conjugate ideals. Here is the key result, which by Corollary 5.3 implies that *all nonzero ideals in \mathcal{O}_K are cancelable*. It will be proved after Theorem 5.9.

Theorem 5.4. *For each ideal \mathfrak{a} in \mathcal{O}_K , the product $\mathfrak{a}\bar{\mathfrak{a}}$ is a principal ideal.*

Up to this point, we have not used Theorem 3.4, which explicitly describes all of the integers of K . For instance, we could have done everything so far in $\mathbf{Z}[\sqrt{d}]$ (defining ideals in $\mathbf{Z}[\sqrt{d}]$ using $\mathbf{Z}[\sqrt{d}]$ -linear combinations, principal ideals, conjugate ideals, and ideal multiplication), even when $\mathcal{O}_K \neq \mathbf{Z}[\sqrt{d}]$, without running into a problem. But to prove Theorem 5.4 we are going to construct some elements about which all we know is that they are in \mathcal{O}_K . If $\mathcal{O}_K \neq \mathbf{Z}[\sqrt{d}]$ then Theorem 5.4 is *false* for $\mathbf{Z}[\sqrt{d}]$.

Example 5.5. When $d \equiv 1 \pmod{4}$, so $\mathcal{O}_K \neq \mathbf{Z}[\sqrt{d}]$, we have the following equality of ideals in $\mathbf{Z}[\sqrt{d}]$:

$$\begin{aligned} (2, 1 + \sqrt{d})(2, 1 + \sqrt{d}) &= (4, 2(1 + \sqrt{d}), d + 1 + 2\sqrt{d}) \\ &= (2)(2, 1 + \sqrt{d}, (d-1)/2 + 1 + \sqrt{d}) \\ &= (2)(2, 1 + \sqrt{d}), \end{aligned}$$

where we can omit the term $(d-1)/2 + 1 + \sqrt{d}$ from the second ideal on the right since it is a \mathbf{Z} -linear combination of 2 and $1 + \sqrt{d}$ (as $(d-1)/2$ is even). If Theorem 5.4 were true for the ideal $(2, 1 + \sqrt{d})$ in the ring $\mathbf{Z}[\sqrt{d}]$ then Corollary 5.3 would imply that this ideal in $\mathbf{Z}[\sqrt{d}]$ is cancelable. But then when we cancel this ideal in the equation $(2, 1 + \sqrt{d})^2 = (2)(2, 1 + \sqrt{d})$ we'd get $(2, 1 + \sqrt{d}) = (2)$, which is false since $1 + \sqrt{d} \notin 2\mathbf{Z}[\sqrt{d}]$.

If we work with the ideal $(2, 1 + \sqrt{d})$ from the bigger ring $\mathcal{O}_K = \mathbf{Z}[(1 + \sqrt{d})/2]$ then the non-cancelable aspect above goes away because in \mathcal{O}_K , $(2, 1 + \sqrt{d}) = (2)(1, (1 + \sqrt{d})/2) = (2)$.

Our approach to Theorem 5.4 relies on the following theorem, which is the heart of the approach and is the first time we need the trace in the context of ideals.

Theorem 5.6. *Let $\mathfrak{a} = (\alpha, \beta)$ be an ideal in \mathcal{O}_K with two generators. Then*

$$\mathfrak{a}\bar{\mathfrak{a}} = (N(\alpha), \text{Tr}(\alpha\bar{\beta}), N(\beta)).$$

Proof. If α or β is 0 then the theorem is easy. We may assume α and β are nonzero.

By a direct computation,

$$\mathfrak{a}\bar{\mathfrak{a}} = (\alpha, \beta)(\bar{\alpha}, \bar{\beta}) = (\alpha\bar{\alpha}, \alpha\bar{\beta}, \beta\bar{\alpha}, \beta\bar{\beta}) = (N(\alpha), \alpha\bar{\beta}, \bar{\alpha}\beta, N(\beta)).$$

We want to show

$$(N(\alpha), \alpha\bar{\beta}, \bar{\alpha}\beta, N(\beta)) = (N(\alpha), \text{Tr}(\alpha\bar{\beta}), N(\beta)).$$

Since $\text{Tr}(\alpha\bar{\beta}) = \alpha\bar{\beta} + \bar{\alpha}\beta$, the ideal on the right is inside the ideal on the left. For the reverse inclusion, we need to show $\alpha\bar{\beta}$ and $\bar{\alpha}\beta$ are in the ideal on the right. We will give the argument for $\alpha\bar{\beta}$, following the treatment in [1, p. 276].

Let g be the greatest common divisor of $N(\alpha)$, $\text{Tr}(\alpha\bar{\beta})$, and $N(\beta)$ in \mathbf{Z} . So g is a factor of all three numbers and, moreover, it is a \mathbf{Z} -linear combination of them. Therefore $(N(\alpha), \text{Tr}(\alpha\bar{\beta}), N(\beta)) = (g)$, so we need to show $\alpha\bar{\beta} \in (g) = g\mathcal{O}_K$. Using the definition of integers of K , we check that $\alpha\bar{\beta}/g$ has trace and norm in \mathbf{Z} . Its trace is

$$\text{Tr}\left(\frac{\alpha\bar{\beta}}{g}\right) = \frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{g} = \frac{\text{Tr}(\alpha\bar{\beta})}{g},$$

which is in \mathbf{Z} since g is a factor of $\text{Tr}(\alpha\bar{\beta})$. Its norm is

$$N\left(\frac{\alpha\bar{\beta}}{g}\right) = \frac{\alpha\bar{\beta}\bar{\alpha}\beta}{g^2} = \frac{N(\alpha)}{g} \frac{N(\beta)}{g},$$

which is in \mathbf{Z} since g is a factor of the norms of α and β . \square

Remark 5.7. It is important in Theorem 5.6 that we work in the ring \mathcal{O}_K . When $d \equiv 1 \pmod{4}$, the formula for $\mathfrak{a}\bar{\mathfrak{a}}$ from the theorem isn't true when $\mathfrak{a} = (2, 1 + \sqrt{d})$ in $\mathbf{Z}[\sqrt{d}]$. We have $\bar{\mathfrak{a}} = (2, 1 - \sqrt{d}) = (2, 1 + \sqrt{d})$ since $1 + \sqrt{d}$ and $1 - \sqrt{d}$ have sum 2, so $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{a}^2 = (2, 1 + \sqrt{d})^2 = (2)\mathfrak{a}$ by Example 5.5. Using $\alpha = 2$ and $\beta = 1 + \sqrt{d}$,

$$(N(\alpha), \text{Tr}(\alpha\bar{\beta}), N(\beta)) = (N(2), \text{Tr}(2 - 2\sqrt{d}), N(1 + \sqrt{d})) = (4, 4, 1 - d) = (4)$$

since $4 \mid (1 - d)$, and $(4) \neq (2)\mathfrak{a}$ since $\mathfrak{a} \neq (2)$. So $\mathfrak{a}\bar{\mathfrak{a}} \neq (N(\alpha), \text{Tr}(\alpha\bar{\beta}), N(\beta))$.

Remark 5.8. Theorem 5.6 makes essential use of the conjugation operation on ideals in \mathcal{O}_K . The integers in higher-degree fields over \mathbf{Q} may not have anything like this operation. For example, in $\mathbf{Q}(\sqrt[3]{2})$, its integers are $\mathbf{Z}[\sqrt[3]{2}] = \mathbf{Z} + \mathbf{Z}\sqrt[3]{2} + \mathbf{Z}\sqrt[3]{4}$ but there is no “conjugation” on elements or on ideals in $\mathbf{Z}[\sqrt[3]{2}]$. There is no simple version of Theorem 5.4 in $\mathbf{Z}[\sqrt[3]{2}]$. If you know Galois theory, there *is* an analogue of Theorem 5.4 for ideals in the integers of a Galois extension of \mathbf{Q} , but for non-Galois extensions like $\mathbf{Q}(\sqrt[3]{2})$ the situation is more complicated.

We can generalize Theorem 5.6 to ideals with more than two generators.

Theorem 5.9. *If the ideal $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ in \mathcal{O}_K has m generators then $\mathfrak{a}\bar{\mathfrak{a}}$ is generated by the m integers $N(\alpha_1), \dots, N(\alpha_m)$ and $\frac{m(m-1)}{2}$ integers $\text{Tr}(\alpha_i\bar{\alpha}_j)$ where $i < j$.*

Proof. The case $m = 1$ is easy. The case $m = 2$ is Theorem 5.6. We will use the case $m = 2$ to handle more generators. By a direct calculation,

$$\begin{aligned} \mathfrak{a}\bar{\mathfrak{a}} &= (\alpha_1, \dots, \alpha_m)(\bar{\alpha}_1, \dots, \bar{\alpha}_m) \\ &= (N(\alpha_1), \dots, N(\alpha_m), \alpha_1\bar{\alpha}_2, \alpha_2\bar{\alpha}_1, \dots, \alpha_i\bar{\alpha}_j, \alpha_j\bar{\alpha}_i, \dots), \end{aligned}$$

where $i < j$. By Theorem 5.6, the four numbers $N(\alpha_i), \alpha_i\bar{\alpha}_j, \alpha_j\bar{\alpha}_i, N(\alpha_j)$ and the three numbers $N(\alpha_i), \text{Tr}(\alpha_i\bar{\alpha}_j), N(\alpha_j)$ are \mathcal{O}_K -linear combinations of each other. Therefore we can replace $\alpha_i\bar{\alpha}_j$ and $\alpha_j\bar{\alpha}_i$ with $\text{Tr}(\alpha_i\bar{\alpha}_j)$ in the generating set for $\mathfrak{a}\bar{\mathfrak{a}}$, and that concludes the proof. \square

Now we can prove Theorem 5.4 very quickly.

Proof. By Theorem 5.9, $\mathfrak{a}\bar{\mathfrak{a}}$ has a set of generators from \mathbf{Z} . Therefore $\mathfrak{a}\bar{\mathfrak{a}}$ is principal by Theorem 4.13. \square

Here is a simple but useful application of Theorem 5.4.

Theorem 5.10. *For ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O}_K , $\mathfrak{a} \mid \mathfrak{b}$ if and only if $\mathfrak{a} \supset \mathfrak{b}$.*

Proof. When $\mathfrak{a} = (0)$, we have $(0) \mid \mathfrak{b} \Leftrightarrow \mathfrak{b} = (0) \Leftrightarrow (0) \supset \mathfrak{b}$. So we may suppose $\mathfrak{a} \neq (0)$. By Theorem 4.29, $\mathfrak{a} \mid \mathfrak{b} \Rightarrow \mathfrak{a} \supset \mathfrak{b}$. Now assume $\mathfrak{a} \supset \mathfrak{b}$. Then $\mathfrak{a}\bar{\mathfrak{a}} \supset \mathfrak{b}\bar{\mathfrak{a}}$. Write $\mathfrak{a}\bar{\mathfrak{a}} = (a)$ (by Theorem 5.4), so $(a) \supset \mathfrak{b}\bar{\mathfrak{a}}$. By Theorem 4.28, $(a) \mid \mathfrak{b}\bar{\mathfrak{a}}$, so $(a)\mathfrak{c} = \mathfrak{b}\bar{\mathfrak{a}}$. Now multiply by \mathfrak{a} : $(a)\mathfrak{c}\mathfrak{a} = \mathfrak{b}(a)$. Cancelling (a) , which is not (0) , gives $\mathfrak{c}\mathfrak{a} = \mathfrak{b}$, so $\mathfrak{a} \mid \mathfrak{b}$. \square

The slogan to remember Theorem 5.10 by is “to contain is to divide.” This is especially useful to keep in mind because the divisibility and inclusion relations are reversed: the factor is the larger ideal.

Corollary 5.11. *The divisors of an ideal \mathfrak{a} in \mathcal{O}_K are precisely the ideals \mathfrak{d} satisfying $\mathfrak{d} \supset \mathfrak{a}$. In particular, $\alpha \in \mathfrak{a}$ if and only if $\mathfrak{a} \mid (\alpha)$.*

Proof. This is immediate from Theorem 5.10. \square

Since the ideals dividing \mathfrak{a} are the ideals containing \mathfrak{a} , it is very easy to create divisors of \mathfrak{a} : if $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and $\alpha \notin \mathfrak{a}$, then $(\alpha_1, \dots, \alpha_m, \alpha)$ is an ideal properly containing \mathfrak{a} , so this ideal is a proper factor of \mathfrak{a} . Of course, if we’re not careful this factor is likely to be (1).

Definition 5.12. The sum of two ideals \mathfrak{a} and \mathfrak{b} is

$$\mathfrak{a} + \mathfrak{b} = \{x + y : x \in \mathfrak{a}, y \in \mathfrak{b}\}.$$

The next theorem shows $\mathfrak{a} + \mathfrak{b}$ is an ideal.

Theorem 5.13. *If $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$ and $\mathfrak{b} = (\beta_1, \dots, \beta_n)$ then*

$$\mathfrak{a} + \mathfrak{b} = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n).$$

Proof. This is left to the reader. Look at the proof of Theorem 4.16 for inspiration if any is needed. \square

Theorem 5.14. *For ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O}_K , the ideal $\mathfrak{a} + \mathfrak{b}$ is a common divisor of \mathfrak{a} and \mathfrak{b} that all other common divisors divide.*

Proof. Since $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} + \mathfrak{b}$ is a divisor of both \mathfrak{a} and \mathfrak{b} because “to contain is to divide.” For an ideal \mathfrak{d} dividing both \mathfrak{a} and \mathfrak{b} we have $\mathfrak{a} \subset \mathfrak{d}$ and $\mathfrak{b} \subset \mathfrak{d}$. As \mathfrak{d} is closed under addition of its elements, we get by Definition 5.12 that $\mathfrak{a} + \mathfrak{b} \subset \mathfrak{d}$, so \mathfrak{d} is a divisor of $\mathfrak{a} + \mathfrak{b}$ by Corollary 5.11. \square

We call $\mathfrak{a} + \mathfrak{b}$ the *greatest common divisor* of \mathfrak{a} and \mathfrak{b} since Theorem 5.14 shows that it has exactly the same feature as the usual greatest common divisor of (positive) integers. Because divisibility of ideals is the same as reverse containment, the greatest common divisor of two ideals is actually the “smallest” ideal containing both \mathfrak{a} and \mathfrak{b} , in the sense of inclusions of ideals.

Example 5.15. Among the ideals of $\mathbf{Z}[\sqrt{-14}]$, the principal ideals (3) and $(1 + \sqrt{-14})$ have greatest common divisor $(3) + (1 + \sqrt{-14}) = (3, 1 + \sqrt{-14})$, which is a non-principal ideal. That is, $\mathfrak{d} \mid (3)$ and $\mathfrak{d} \mid (1 + \sqrt{-14})$ if and only if $\mathfrak{d} \mid (3, 1 + \sqrt{-14})$.

Since every ideal in \mathcal{O}_K has the form $(\alpha_1, \dots, \alpha_m)$, every ideal in \mathcal{O}_K is the greatest common divisor of principal ideals because

$$(\alpha_1, \dots, \alpha_m) = \alpha_1 \mathcal{O}_K + \dots + \alpha_m \mathcal{O}_K = (\alpha_1) + \dots + (\alpha_m).$$

This shows how the principal ideals “control” all the ideals in \mathcal{O}_K .

6. IDEAL NORMS

For a nonzero ideal \mathfrak{a} , Theorem 5.9 says $\mathfrak{a}\bar{\mathfrak{a}}$ is generated by elements of \mathbf{Z} , so it is principal with a generator in \mathbf{Z} . The generator can be chosen in \mathbf{Z}^+ by changing its sign if necessary. (Easily $(a) = (-a)$.) This generator is unique, because if a and b are in \mathbf{Z}^+ and $(a) = (b)$ as ideals in \mathcal{O}_K then $a = bu$ for some $u \in \mathcal{O}_K^\times \cap \mathbf{Q} = \{\pm 1\}$ (Theorem 3.8), so $u = 1$ since a and b are positive.

Definition 6.1. For a nonzero ideal \mathfrak{a} in \mathcal{O}_K , set $N\mathfrak{a}$ to be the positive integer that generates $\mathfrak{a}\bar{\mathfrak{a}}$:

$$\mathfrak{a}\bar{\mathfrak{a}} = (N\mathfrak{a}), \quad N\mathfrak{a} \in \mathbf{Z}^+.$$

We call $N\mathfrak{a}$ the (ideal) *norm* of \mathfrak{a} .

Example 6.2. By (4.3), in $\mathbf{Z}[\sqrt{-14}]$ the ideal $(3, 1 + \sqrt{-14})$ has norm 3.

Theorem 6.3. *The ideal norm on \mathcal{O}_K is compatible with the element norm on principal ideals: if $\mathfrak{a} = (\alpha)$ then $N\mathfrak{a} = |N(\alpha)|$.*

Proof. We have $\mathfrak{a}\bar{\mathfrak{a}} = (\alpha)(\bar{\alpha}) = (\alpha\bar{\alpha}) = (N(\alpha)) = (|N(\alpha)|)$. This equals $(N\mathfrak{a})$, so $N\mathfrak{a} = |N(\alpha)|$ since both are positive integers. \square

Theorem 6.4. *For nonzero ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O}_K , $N(\mathfrak{a}\mathfrak{b}) = N\mathfrak{a}N\mathfrak{b}$.*

Proof. We have

$$(N(\mathfrak{a}\mathfrak{b})) = \mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}\mathfrak{b}} = \mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}\bar{\mathfrak{b}} = \mathfrak{a}\bar{\mathfrak{a}}\mathfrak{b}\bar{\mathfrak{b}} = (N\mathfrak{a})(N\mathfrak{b}) = (N\mathfrak{a}N\mathfrak{b}).$$

Therefore the positive integers $N(\mathfrak{a}\mathfrak{b})$ and $N\mathfrak{a}N\mathfrak{b}$ each generate the same principal ideal in \mathcal{O}_K , so they are equal. \square

Corollary 6.5. *For nonzero ideals \mathfrak{a} and \mathfrak{b} in \mathcal{O}_K , if $\mathfrak{a} \mid \mathfrak{b}$ then $N\mathfrak{a} \mid N\mathfrak{b}$ in \mathbf{Z} .*

Proof. Write $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ and take norms of both sides. \square

The converse of Corollary 6.5 is false: The ideals $\mathfrak{a} = (1 + \sqrt{-14})$ and $\mathfrak{b} = (1 - \sqrt{-14})$ in $\mathbf{Z}[\sqrt{-14}]$ have equal norm but \mathfrak{a} does not divide \mathfrak{b} .

Note $N\mathfrak{a} = 1$ if and only if $\mathfrak{a} = (1)$. In one direction, it is trivial that $N((1)) = 1$. Conversely, if $N\mathfrak{a} = 1$ then $\mathfrak{a}\bar{\mathfrak{a}} = (1)$, so $\mathfrak{a} \mid (1)$. Therefore $\mathfrak{a} \supset (1) = \mathcal{O}_K$, so $\mathfrak{a} = (1)$. Thus each ideal $\mathfrak{a} \neq (1)$ has $N\mathfrak{a} > 1$. This will be important later when we prove theorems about ideal factorization by induction on the ideal norm.

We did not define the norm of the ideal (0) , but it is perfectly natural to set $N((0)) = 0$. All results about ideal norms so far now extend to the zero ideal, but the next property is specific to nonzero ideals.

Corollary 6.6. *For a nonzero ideal \mathfrak{a} in \mathcal{O}_K , every ideal factor of \mathfrak{a} other than \mathfrak{a} has norm less than $N\mathfrak{a}$.*

Proof. Let \mathfrak{b} be a factor of \mathfrak{a} other than \mathfrak{a} , so $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$ and $\mathfrak{c} \neq (1)$. Since $N\mathfrak{a} = N\mathfrak{b}N\mathfrak{c}$ with $N\mathfrak{a} \neq (0)$ and $N\mathfrak{c} > 1$, $N\mathfrak{b} < N\mathfrak{a}$. \square

In practice, how do we compute the norm of an ideal \mathfrak{a} in \mathcal{O}_K ? Theorem 5.9 provides an algorithm: for a set of generators for \mathfrak{a} , compute the greatest common divisor (in \mathbf{Z}) of the norms and “cross-traces” of the generators as described in the statement of Theorem 5.9. That number is the norm. Let’s look at some examples in $\mathbf{Z}[\sqrt{-14}]$.

Example 6.7. Let $\mathfrak{a} = (3, 1 + \sqrt{-14})$. The ideal $\mathfrak{a}\bar{\mathfrak{a}}$ is generated by $N(3)$, $\text{Tr}(3(1 - \sqrt{-14}))$, and $N(1 + \sqrt{-14})$, which are 9, 6, and 15. Their greatest common divisor is 3, so we get $N(3, 1 + \sqrt{-14}) = 3$.

Example 6.8. Let $\mathfrak{a} = (1 + \sqrt{-14}, 1 - \sqrt{-14})$. The norm is the greatest common divisor of $N(1 \pm \sqrt{-14}) = 15$ and $\text{Tr}((1 + \sqrt{-14})^2) = -26$. Since 15 and -26 are relatively prime, $N\mathfrak{a} = 1$. Therefore $\mathfrak{a} = (1)$. Notice in particular that $N\mathfrak{a}$ is *not* the greatest common divisor of the norms of a set of generators; we needed the trace term as well.

Example 6.9. Let $\mathfrak{a} = (4 + \sqrt{-14}, 2 - \sqrt{-14})$. From $N(4 + \sqrt{-14}) = 30$, $N(2 - \sqrt{-14}) = 18$, and $\text{Tr}((4 + \sqrt{-14})(2 + \sqrt{-14})) = -12$, we get $N\mathfrak{a} = 6$.

7. PRIME IDEALS AND UNIQUE FACTORIZATION

Definition 7.1. An ideal \mathfrak{p} in \mathcal{O}_K is called *prime* if $\mathfrak{p} \neq (1)$ and \mathfrak{p} can't be written as a product $\mathfrak{a}\mathfrak{b}$ unless $\mathfrak{a} = (1)$ or $\mathfrak{b} = (1)$.

Note the ideal (0) is not considered prime.¹ Here is a criterion to recognize a prime ideal.

Theorem 7.2. *An ideal in \mathcal{O}_K whose norm is prime in \mathbf{Z} is a prime ideal.*

Proof. Let $N\mathfrak{a} = p$ be prime. If $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$, then taking norms shows $p = N\mathfrak{b}N\mathfrak{c}$. Since p is prime, either \mathfrak{b} or \mathfrak{c} has norm 1, so either \mathfrak{b} or \mathfrak{c} is (1) . \square

Example 7.3. In $\mathbf{Z}[\sqrt{-14}]$, the ideal $(3, 1 + \sqrt{-14})$ has norm 3, so it is a prime ideal. Similarly, the ideals $(3, 1 - \sqrt{-14})$, $(5, 1 + \sqrt{-14})$, and $(5, 1 - \sqrt{-14})$ are prime.

The converse to Theorem 7.2 is false: an ideal can be prime without having prime norm.

Example 7.4. In $\mathbf{Z}[\sqrt{-14}]$, we will show the ideal (11) , whose norm is 121, is prime. Assume $(11) = \mathfrak{a}\mathfrak{b}$ with $\mathfrak{a} \neq (1)$ and $\mathfrak{b} \neq (1)$. Then taking norms implies $121 = N\mathfrak{a}N\mathfrak{b}$, so $N\mathfrak{a} = 11$. Write $\mathfrak{a} = (\alpha_1, \dots, \alpha_m)$. Since $\alpha_i \in \mathfrak{a}$ we have $\mathfrak{a} \mid (\alpha_i)$ by Theorem 5.10, so taking norms gives $11 \mid N(\alpha_i)$. Which elements of $\mathbf{Z}[\sqrt{-14}]$ have norm divisible by 11?

If $x + y\sqrt{-14}$ satisfies $x^2 + 14y^2 \equiv 0 \pmod{11}$ then $x^2 \equiv -3y^2 \pmod{11}$. Since $-3 \pmod{11}$ is not a square we must have $y \equiv 0 \pmod{11}$ and then $x \equiv 0 \pmod{11}$. That implies $x + y\sqrt{-14}$ is divisible by 11 in $\mathbf{Z}[\sqrt{-14}]$.

Returning to the setup where $(11) = \mathfrak{a}\mathfrak{b}$, the previous paragraph implies that each element of \mathfrak{a} is a multiple of 11. Factoring 11 from each generator gives $\mathfrak{a} = (11)\mathfrak{c}$ for some ideal \mathfrak{c} . But then $N\mathfrak{a}$ is divisible by 121, while $N\mathfrak{a} = 11$. This is a contradiction, so (11) is a prime ideal in $\mathbf{Z}[\sqrt{-14}]$.

Theorem 7.5. *If an ideal in \mathcal{O}_K is prime then its conjugate ideal is prime.*

Proof. Factorizations of an ideal and factorizations of its conjugate ideal can be carried into each other by taking conjugates, with non-unit ideals having non-unit conjugates. \square

To prove unique factorization in the positive integers, there are three steps:

- show prime numbers satisfy the property $p \mid ab \Rightarrow p \mid a$ or $p \mid b$ in \mathbf{Z} ,
- show by induction that every positive integer > 1 has a prime factorization,
- show by induction that the prime factorization is unique, using the first step and cancellation to reduce to a smaller case.

¹For readers who know more ring theory, the ideal (0) in \mathcal{O}_K should be prime by the standard definition of prime ideals in abstract algebra. We avoid that in our definition of prime ideals to keep things simpler.

The following theorem is the analogue of the first step above. The corresponding formulation for irreducible elements is false (Example 3.16).

Theorem 7.6. *If \mathfrak{p} is a prime ideal in \mathcal{O}_K and $\mathfrak{p} \mid \mathfrak{a}\mathfrak{b}$ then $\mathfrak{p} \mid \mathfrak{a}$ or $\mathfrak{p} \mid \mathfrak{b}$.*

Proof. We will assume \mathfrak{p} does not divide \mathfrak{a} and prove $\mathfrak{p} \mid \mathfrak{b}$. The ideal $\mathfrak{p} + \mathfrak{a}$ is a common divisor of \mathfrak{p} and \mathfrak{a} (Theorem 5.14). The only divisors of \mathfrak{p} are \mathfrak{p} and (1) by our definition of prime ideals in \mathcal{O}_K . Because \mathfrak{p} does not divide \mathfrak{a} , $\mathfrak{p} + \mathfrak{a} \neq \mathfrak{p}$. Therefore $\mathfrak{p} + \mathfrak{a} = (1)$, so $1 = x + \alpha$ for some $x \in \mathfrak{p}$ and $\alpha \in \mathfrak{a}$. Then for each $\beta \in \mathfrak{b}$,

$$\beta = 1 \cdot \beta = x\beta + \alpha\beta \in \mathfrak{p} + \mathfrak{a}\mathfrak{b} \subset \mathfrak{p},$$

which shows $\mathfrak{b} \subset \mathfrak{p}$. Thus $\mathfrak{p} \mid \mathfrak{b}$ (Corollary 5.11). \square

Corollary 7.7. *If \mathfrak{p} is prime in \mathcal{O}_K and $\mathfrak{p} \mid \mathfrak{a}_1 \cdots \mathfrak{a}_r$ then $\mathfrak{p} \mid \mathfrak{a}_i$ for some i .*

Proof. Induct on r . \square

Now we work out the analogue of the second step towards unique factorization.

Theorem 7.8. *Every nonzero ideal $\neq (1)$ in \mathcal{O}_K admits a prime ideal factorization.*

Proof. Mimic the proof of Theorem 3.15 by using induction on the ideal norm. \square

Theorem 7.9. *The prime ideal factorization of a nonzero proper ideal in \mathcal{O}_K is unique up to the order of the factors. That is, for each nonzero $\mathfrak{a} \neq (1)$, if*

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r = \mathfrak{q}_1 \cdots \mathfrak{q}_s$$

where the \mathfrak{p}_i 's and \mathfrak{q}_j 's are prime, then the number of prime ideals in both factorizations is the same and $\mathfrak{p}_i = \mathfrak{q}_i$ after a suitable relabelling of the indices.

Proof. We argue by induction on the norm of the ideal. A prime ideal has no factorization into a product of prime ideals except itself, so unique factorization is settled for prime ideals. This includes ideals with norm 2 (Theorem 7.2). For $n \geq 3$, suppose all ideals with norm from 2 to $n - 1$ have unique prime ideal factorization. Let \mathfrak{a} be an ideal with norm n and two prime ideal factorizations:

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r = \mathfrak{q}_1 \cdots \mathfrak{q}_s.$$

Here we may take $r > 1$ and $s > 1$ since we may suppose \mathfrak{a} is not a prime ideal.

Because $\mathfrak{p}_1 \mid \mathfrak{a}$ we can say $\mathfrak{p}_1 \mid \mathfrak{q}_1 \cdots \mathfrak{q}_s$. By Corollary 7.7, \mathfrak{p}_1 divides some \mathfrak{q}_j . Since ideal multiplication is commutative, we can relabel the \mathfrak{q}_j 's so that $\mathfrak{p}_1 \mid \mathfrak{q}_1$. Then since \mathfrak{q}_1 is prime we must have $\mathfrak{p}_1 = \mathfrak{q}_1$ (since $\mathfrak{p}_1 \neq (1)$). Because all nonzero ideals in \mathcal{O}_K are cancelable, we can remove \mathfrak{p}_1 from the two prime factorizations:

$$\mathfrak{p}_2 \cdots \mathfrak{p}_r = \mathfrak{q}_2 \cdots \mathfrak{q}_s.$$

This is a prime ideal factorization of an ideal with norm $N\mathfrak{a}/N\mathfrak{p}_1 < N\mathfrak{a} = n$, so the inductive hypothesis applies: the number of prime ideals on both sides is the same (so $r - 1 = s - 1$, hence $r = s$) and after a suitable relabelling $\mathfrak{p}_i = \mathfrak{q}_i$ for $i = 2, \dots, r$. We have $\mathfrak{p}_1 = \mathfrak{q}_1$ already, so we are done. \square

We will work out examples of prime ideal factorizations in Section 9.

Remark 7.10. That we prove theorems about ideals by induction on their norm does not actually mean every positive integer is the norm of some ideal. For instance, there are no ideals in $\mathbf{Z}[\sqrt{-14}]$ with norm 11, as we saw in Example 7.4. That means some cases of these induction arguments are actually empty cases.

The next theorem shows that non-principal ideals in \mathcal{O}_K are the obstruction to unique factorization of elements in \mathcal{O}_K .

Theorem 7.11. *There is unique factorization of elements of \mathcal{O}_K if and only if every ideal in \mathcal{O}_K is principal.*

Proof. First suppose \mathcal{O}_K has unique factorization of elements.

Step 1: For each irreducible π in \mathcal{O}_K , the principal ideal (π) is prime.

Let \mathfrak{a} be an ideal dividing (π) , so $\mathfrak{a} \supset (\pi)$. We want to show \mathfrak{a} is (1) or (π) . Suppose $\mathfrak{a} \neq (\pi)$, so there is an $\alpha \in \mathfrak{a}$ with $\alpha \notin (\pi)$. Writing $(\pi) = \mathfrak{a}\mathfrak{b}$, $\mathfrak{b} \mid (\pi)$ and for every $\beta \in \mathfrak{b}$ we have $\alpha\beta \in \mathfrak{a}\mathfrak{b} = (\pi)$, so $\pi \mid \alpha\beta$. By unique factorization of elements, π is an irreducible factor of either α or β . Since π does not divide α , we must have $\pi \mid \beta$, so $\beta \in (\pi)$. This holds for all $\beta \in \mathfrak{b}$, so $\mathfrak{b} \subset (\pi)$. Since $\mathfrak{b} \mid (\pi)$ and $(\pi) \mid \mathfrak{b}$, $\mathfrak{b} = (\pi)$. Therefore $(\pi) = \mathfrak{a}(\pi)$, so $\mathfrak{a} = (1)$.

Step 2: Every prime ideal in \mathcal{O}_K is principal.

Let \mathfrak{p} be a prime ideal, so $\mathfrak{p} \neq (0)$ by our definition of prime ideals in \mathcal{O}_K . Then $\mathfrak{p} \mid (a)$ for some nonzero $a \in \mathbf{Z}$, such as $N\mathfrak{p}$. Factor a into irreducibles in \mathcal{O}_K (Theorem 3.15), say $a = \pi_1 \cdots \pi_r$. Then $(a) = (\pi_1) \cdots (\pi_r)$, so \mathfrak{p} divides some (π_i) . Since (π_i) is prime by Step 1, $\mathfrak{p} = (\pi_i)$.

Step 3: Every ideal in \mathcal{O}_K is principal.

The zero ideal is obviously principal. Each nonzero ideal is a product of prime ideals, which are principal by Step 2, so their product is principal.

This concludes the “only if” direction.

Now assume every ideal in \mathcal{O}_K is principal. We want to show \mathcal{O}_K has unique factorization of elements. The existence of factorization into irreducibles is Theorem 3.15. To get uniqueness, we just need to show for each irreducible π that when $\pi \mid \alpha\beta$ in \mathcal{O}_K either $\pi \mid \alpha$ or $\pi \mid \beta$. (The analogue of this property for prime ideals in Theorem 7.6 was used to prove uniqueness of prime ideal factorizations.) Suppose $\pi \mid \alpha\beta$ and π does not divide α . We want to show $\pi \mid \beta$. The only factors of π are units and unit multiples of π , so the only common factors of π and α are units. The ideal (π, α) is principal by hypothesis, say $(\pi, \alpha) = (\delta)$, so δ is a common factor of π and α . Thus δ is a unit, so $(\pi, \alpha) = (1)$. That means $\pi x + \alpha y = 1$ for some x and y in \mathcal{O}_K . Multiplying through by β , $\pi\beta x + \alpha\beta y = \beta$. Since $\pi \mid \alpha\beta$, we conclude that $\pi \mid \beta$. \square

8. CONSTRUCTING PRIME IDEALS

We have met examples of prime ideals, like $(3, 1 + \sqrt{-14})$ and (11) in $\mathbf{Z}[\sqrt{-14}]$. What does a general prime ideal in \mathcal{O}_K look like? We will describe them in terms of prime numbers.

Theorem 8.1. *Every prime ideal in \mathcal{O}_K divides a unique prime number. That is, if \mathfrak{p} is a prime ideal then $\mathfrak{p} \mid (p)$ for one prime p in \mathbf{Z}^+ .*

Proof. The ideal $\mathfrak{p}\bar{\mathfrak{p}} = (N\mathfrak{p})$ is divisible by \mathfrak{p} and has a generator in \mathbf{Z}^+ . Since $\mathfrak{p} \neq (1)$, $N\mathfrak{p} > 1$. Factor $N\mathfrak{p}$ into primes in \mathbf{Z}^+ , say

$$N\mathfrak{p} = p_1 p_2 \cdots p_r.$$

Then $\mathfrak{p}\bar{\mathfrak{p}} = (p_1 p_2 \cdots p_r) = (p_1) \cdots (p_r)$, so \mathfrak{p} divides some (p_i) by Corollary 7.7.

For the uniqueness, assume $\mathfrak{p} \mid (p)$ and $\mathfrak{p} \mid (q)$ for two different prime numbers p and q . Then $p \in \mathfrak{p}$ and $q \in \mathfrak{p}$. Since p and q are relatively prime, \mathfrak{p} contains a pair of relatively prime integers, so $\mathfrak{p} = (1)$. This is a contradiction. \square

Corollary 8.2. *Every prime ideal in \mathcal{O}_K has norm p or p^2 for a prime number p .*

Proof. Let \mathfrak{p} be a prime ideal in \mathcal{O}_K . Then there is a prime number p such that $\mathfrak{p} \mid (p)$. Taking ideal norms, $N\mathfrak{p} \mid N((p))$. Since $N((p)) = |N(p)| = p^2$, $N\mathfrak{p}$ is p or p^2 . \square

There are two ways to characterize the unique prime number p such that $\mathfrak{p} \mid (p)$: p is the only prime number in \mathfrak{p} and $N\mathfrak{p}$ is a power of p .

The importance of Theorem 8.1 is that it says we can discover every prime ideal in \mathcal{O}_K by factoring prime numbers in \mathcal{O}_K . For instance, in $\mathbf{Z}[\sqrt{-14}]$ we know $(2) = (2, \sqrt{-14})^2$ and $(3) = (3, 1 + \sqrt{-14})(3, 1 - \sqrt{-14})$. Therefore $(2, \sqrt{-14})$ is the only prime ideal with 2-power norm and $(3, 1 + \sqrt{-14})$ and $(3, 1 - \sqrt{-14})$ are the only prime ideals with 3-power norm. (By Example 4.25, $(3, 1 + \sqrt{-14}) \neq (3, 1 - \sqrt{-14})$.)

The next theorem shows how each prime number (really, the principal ideal generated by each prime number) factors in \mathcal{O}_K , and thus shows what the prime ideals of \mathcal{O}_K look like.

Theorem 8.3. *Let $K = \mathbf{Q}[\sqrt{d}]$ be a quadratic field with squarefree d and $\mathcal{O}_K = \mathbf{Z}[\omega]$, with $f(X)$ the quadratic polynomial having ω and $\bar{\omega}$ as roots:*

$$f(X) = \begin{cases} X^2 - d, & \text{if } d \not\equiv 1 \pmod{4}, \\ X^2 - X + \frac{1-d}{4}, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

For each prime number p , how (p) factors in \mathcal{O}_K matches how $f(X)$ factors modulo p :

- (1) *If $f(X) \pmod{p}$ is irreducible then (p) is prime in \mathcal{O}_K with norm p^2 .*
- (2) *If $f(X) \equiv (X - c)(X - c') \pmod{p}$ with $c \not\equiv c' \pmod{p}$ then $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ where $\mathfrak{p} \neq \bar{\mathfrak{p}}$ and the conjugate ideals \mathfrak{p} and $\bar{\mathfrak{p}}$ have norm p .*
- (3) *If $f(X) \equiv (X - c)^2 \pmod{p}$ then $(p) = \mathfrak{p}^2$ and $N\mathfrak{p} = p$.*

In particular, prime ideals in \mathcal{O}_K have prime norm except for principal primes (p) where p is a prime number such that $f(X) \pmod{p}$ is irreducible.

Note the exponent of p in the norms of prime ideals dividing (p) matches the degrees of the irreducible factors of $f(X) \pmod{p}$.

The statement of Theorem 8.3 is more important than our proof below, which is tedious and computation-dependent. With better techniques (going beyond the level at which this is written) the proof can be streamlined.

Proof. So that we have a uniform notation, we will write $f(X)$ as $X^2 - tX + n$, where $t = \text{Tr}(\omega)$ and $n = N(\omega)$. In particular, $\omega^2 - t\omega + n = 0$, so $\omega^2 = t\omega - n$. For $a + b\omega$ in $\mathbf{Z}[\omega]$, $N(a + b\omega) = a^2 + tab + nb^2$.

(1): To prove that irreducibility of $f(X) \pmod{p}$ implies (p) is prime, we will prove the contrapositive: if (p) is not prime in \mathcal{O}_K then $f(X) \pmod{p}$ has a root, so is reducible.

Since (p) has norm p^2 , a proper ideal factor of (p) that is not (1) has norm p , and therefore is a prime ideal. So if (p) is not prime then there is a prime ideal \mathfrak{p} with norm p and $\mathfrak{p} \mid (p)$. Then $p \in \mathfrak{p}$ and $\mathfrak{p} \neq (p)$, so \mathfrak{p} properly contains (p) . Pick an element of \mathfrak{p} not in (p) , say $a + b\omega$. That is, $a + b\omega \in \mathfrak{p}$ and $a + b\omega \notin (p)$. Since $\mathfrak{p} \mid (a + b\omega)$, $p \mid N(a + b\omega)$, so

$$(8.1) \quad a^2 + tab + nb^2 \equiv 0 \pmod{p}.$$

The number b is not divisible by p , by contradiction: if $b \equiv 0 \pmod{p}$ then $a^2 \equiv 0 \pmod{p}$, so both a and b are multiples of p . Then $a + b\omega \in (p)$, which is not true. Since $b \not\equiv 0 \pmod{p}$, we can divide (8.1) by b to get $c^2 + tc + n \equiv 0 \pmod{p}$ where $c \equiv ab^{-1} \pmod{p}$. Since

$f(X) = X^2 - tX + n$, we see that $f(-c) \equiv 0 \pmod{p}$, so $f(X) \pmod{p}$ is reducible. This completes the proof that if $f(X) \pmod{p}$ is irreducible then (p) is a prime ideal.

(2): Now assume $f(X) \equiv (X - c)(X - c') \pmod{p}$, where $c \not\equiv c' \pmod{p}$. Set $\mathfrak{p} = (p, \omega - c)$. We will show \mathfrak{p} is prime with norm p and $\mathfrak{p}\bar{\mathfrak{p}} = (p)$.

Since $p \in \mathfrak{p}$, $\mathfrak{p} \mid (p)$. We have $\mathfrak{p} \neq (p)$ since $\omega - c \notin (p)$. Therefore $N\mathfrak{p}$ properly divides p^2 , so $N\mathfrak{p}$ is 1 or p . The norm of \mathfrak{p} is the greatest common divisor of $N(p) = p^2$, $\text{Tr}(p(\bar{\omega} - c)) = p(t - 2c)$, and $N(\omega - c) = f(c)$, which are all multiples of p . Therefore $p \mid N\mathfrak{p}$, so $N\mathfrak{p} = p$ and \mathfrak{p} is a prime ideal.

Now we compute $\mathfrak{p}\bar{\mathfrak{p}}$:

$$\begin{aligned} \mathfrak{p}\bar{\mathfrak{p}} &= (p, \omega - c)(p, \bar{\omega} - c) \\ &= (p^2, p(\omega - c), p(\bar{\omega} - c), N(\omega - c)) \\ &= (p^2, p(\omega - c), p(\bar{\omega} - c), f(c)). \end{aligned}$$

Since $f(c) \equiv 0 \pmod{p}$, the ratio $f(c)/p$ is in \mathbf{Z} . Therefore

$$\mathfrak{p}\bar{\mathfrak{p}} = (p)(p, \omega - c, \bar{\omega} - c, f(c)/p).$$

We want to show the second ideal on the right is (1) . Both ideals on the left have norm p and (p) has norm p^2 , so the second ideal on the right must have norm 1, so the ideal is (1) .

The last thing we need to check in this second case is that $\mathfrak{p} \neq \bar{\mathfrak{p}}$. Assume otherwise, so $(p) = \mathfrak{p}^2$. We compute \mathfrak{p}^2 explicitly:

$$\begin{aligned} \mathfrak{p}^2 &= (p, \omega - c)^2 \\ &= (p^2, p(\omega - c), (\omega - c)^2) \\ &= (p^2, p(\omega - c), \omega^2 - 2c\omega + c^2) \\ &= (p^2, p(\omega - c), (t - 2c)\omega - n + c^2). \end{aligned}$$

For this to be (p) requires each generator be divisible by p . Therefore we need $t \equiv 2c \pmod{p}$. Since $t \equiv c + c' \pmod{p}$ (compare the coefficient of X on both sides of $f(X) \equiv (X - c)(X - c') \pmod{p}$), we obtain $c' \equiv c \pmod{p}$, and that contradicts the distinctness of the roots of $f(X) \pmod{p}$. So $\mathfrak{p} \neq \bar{\mathfrak{p}}$.

(3): Set $\mathfrak{p} = (p, \omega - c)$. We want to show \mathfrak{p} is a prime ideal with norm p and $(p) = \mathfrak{p}^2$.

Arguing as in the second case, $N\mathfrak{p}$ is p , so \mathfrak{p} is prime. Now compute \mathfrak{p}^2 :

$$\begin{aligned} \mathfrak{p}^2 &= (p, \omega - c)(p, \omega - c) \\ &= (p^2, p(\omega - c), (\omega - c)^2) \\ &= (p^2, p(\omega - c), \omega^2 - 2c\omega + c^2). \end{aligned}$$

Since $\omega^2 = t\omega - n$,

$$(8.2) \quad \mathfrak{p}^2 = (p^2, p(\omega - c), (t - 2c)\omega + c^2 - n).$$

Since $f(X) \equiv (X - c)^2 \pmod{p}$, equating coefficients on both sides shows $t \equiv 2c \pmod{p}$ and $n \equiv c^2 \pmod{p}$. Thus we can factor (p) from \mathfrak{p}^2 in (8.2):

$$\mathfrak{p}^2 = (p) \left(p, \omega - c, \frac{t - 2c}{p}\omega + \frac{c^2 - n}{p} \right).$$

Taking the norms of both sides shows the second ideal on the right has norm 1, so the ideal is (1) . \square

Example 8.4. How does (2) factor in the integers of $\mathbf{Q}[\sqrt{-39}]$? Although $X^2 + 39 \equiv (X + 1)^2 \pmod{2}$, it is incorrect to conclude that $(2) = \mathfrak{p}^2$ because the integers of $\mathbf{Q}[\sqrt{-39}]$ are not $\mathbf{Z}[\sqrt{-39}]$. The relevant polynomial is not $X^2 + 39$, but rather $X^2 - X + 10$ (which has $(1 + \sqrt{-39})/2$ as a root). Since $X^2 - X + 10 \equiv X(X - 1) \pmod{2}$, the correct factorization of (2) is $\mathfrak{p}\bar{\mathfrak{p}}$.

When $p \neq 2$, how the quadratic polynomial $f(X)$ factors modulo p is determined by the discriminant: there are two different roots if the discriminant is a nonzero square mod p , no roots if the discriminant is not a square mod p , and a repeated root if the discriminant is $0 \pmod{p}$. The two formulas for $f(X)$ are $X^2 - d$ and $X^2 - X + \frac{1-d}{4}$, which have respective discriminants $4d$ and d , so how $f(X) \pmod{p}$ factors is determined by the Legendre symbol $(\frac{d}{p})$. (This Legendre symbol is 0 when $p \mid d$.) What can we say when $p = 2$? The way $f(X) \pmod{2}$ factors depends on $d \pmod{8}$. We can translate the results of Theorem 8.3 into simple formulas based on $(\frac{d}{p})$ and $d \pmod{8}$, which we record in Tables 2 and 3. The second column in each table describes how (p) factors into prime ideals in the integers of $\mathbf{Q}[\sqrt{d}]$. In both tables, we write $\bar{\mathfrak{p}}$ when $\mathfrak{p} \neq \bar{\mathfrak{p}}$. In Table 2, $c \pmod{p}$ is a root of $f(X) \pmod{p}$.

$(\frac{d}{p})$	(p)	\mathfrak{p}
1	$\mathfrak{p}\bar{\mathfrak{p}}$	$(p, \omega - c)$
-1	\mathfrak{p}	(p)
0	\mathfrak{p}^2	$(p, \omega - c)$

TABLE 2. $p \neq 2$

$d \pmod{8}$	(2)	\mathfrak{p}
1	$\mathfrak{p}\bar{\mathfrak{p}}$	$(2, \frac{1+\sqrt{d}}{2})$
5	\mathfrak{p}	(2)
3, 7	\mathfrak{p}^2	$(2, \sqrt{d} - 1)$
even	\mathfrak{p}^2	$(2, \sqrt{d})$

TABLE 3. $p = 2$

Although multiplication of ideals increases the number of generators (an ideal with m generators times an ideal with n generators is an ideal with mn generators), in practice every ideal we have met can be expressed in terms of two generators regardless of how many generators it starts with. Consider Example 4.17, for instance. And each prime ideal in \mathcal{O}_K in Tables 2 and 3 requires at most two generators. This is actually true of all ideals in \mathcal{O}_K .

Theorem 8.5. *Every ideal in \mathcal{O}_K needs at most two generators. More precisely, if \mathfrak{a} is a nonzero ideal and α is an arbitrary nonzero element of \mathfrak{a} , then $\mathfrak{a} = (\alpha, \beta)$ for a suitably chosen $\beta \in \mathfrak{a}$.*

The theorem says one generator can be chosen arbitrarily among the nonzero elements of the ideal. That choice affects the second generator in the description of the ideal. Theorem 8.5 can be proved using an analogue of the Chinese remainder theorem for \mathcal{O}_K or using the structure of finitely generated abelian groups. We omit a proof.

9. WORKED EXAMPLES

We will compute some prime ideal factorizations in $\mathbf{Z}[\sqrt{-14}]$. First we specialize Tables 2 and 3 to small primes in $\mathbf{Z}[\sqrt{-14}]$. See Table 4.

p	(p)	\mathfrak{p}
2	\mathfrak{p}_2^2	$\mathfrak{p}_2 = (2, \sqrt{-14})$
3	$\mathfrak{p}_3\bar{\mathfrak{p}}_3$	$\mathfrak{p}_3 = (3, \sqrt{-14} + 1)$
5	$\mathfrak{p}_5\bar{\mathfrak{p}}_5$	$\mathfrak{p}_5 = (5, \sqrt{-14} + 1)$
7	\mathfrak{p}_7^2	$\mathfrak{p}_7 = (7, \sqrt{-14})$
11	(11)	(11)
13	$\mathfrak{p}_{13}\bar{\mathfrak{p}}_{13}$	$\mathfrak{p}_{13} = (13, \sqrt{-14} + 5)$
17	(17)	(17)
19	$\mathfrak{p}_{19}\bar{\mathfrak{p}}_{19}$	$\mathfrak{p}_{19} = (19, \sqrt{-14} + 9)$
23	$\mathfrak{p}_{23}\bar{\mathfrak{p}}_{23}$	$\mathfrak{p}_{23} = (\sqrt{-14} + 3)$

TABLE 4. Factoring (p) in $\mathbf{Z}[\sqrt{-14}]$

The way (p) factors in $\mathbf{Z}[\sqrt{-14}]$ is determined by the way $X^2 + 14$ factors mod p , and that is determined by $(\frac{-14}{p})$ if $p \neq 2$. If $(\frac{-14}{p}) = 1$ then (p) is a product of two different (conjugate) ideals with norm p . When $-14 \equiv c^2 \pmod{p}$, one of the prime factors \mathfrak{p} is $(p, \sqrt{-14} - c)$. If $(\frac{-14}{p}) = -1$ then (p) is a prime ideal with norm p^2 . If $p = 2$ or 7 then (p) is the square of a prime ideal with norm p . The first odd p such that $(\frac{-14}{p}) = 1$ are 3, 5, 13, and 19. Therefore we have the factorizations in Table 4, where we leave a principal ideal unfactored if it stays prime (*e.g.*, $p = 11$). The last ideal, \mathfrak{p}_{23} , should be $(23, \sqrt{-14} + 3)$ according to the general methods for computing prime ideals. But in this particular case $23 = (3 + \sqrt{-14})(3 - \sqrt{-14})$, so we can drop 23 as a generator of the ideal. The prime ideal factors of 2, 3, 5, 7, 13, and 19 in the table are non-principal since the equation $x^2 + 14y^2 = p$ has no solution for these values of p .

With this information we can start factoring ideals with generators not in \mathbf{Z} . The central idea is to compute the norm of the ideal and use our knowledge of prime factorization of the norm in \mathbf{Z} to help us. Remember that prime ideals only have prime-power norm.

Example 9.1. We will factor $(1 + \sqrt{-14})$. The ideal has norm 15. Writing $(1 + \sqrt{-14})$ as a product of prime ideals, the product of their norms is 15, so $(1 + \sqrt{-14})$ must be the product of a prime ideal of norm 3 and a prime ideal of norm 5. Since $1 + \sqrt{-14} \in \mathfrak{p}_3$ and $1 + \sqrt{-14} \in \mathfrak{p}_5$, $(1 + \sqrt{-14})$ is divisible by \mathfrak{p}_3 and \mathfrak{p}_5 . Hence $(1 + \sqrt{-14}) = \mathfrak{p}_3\mathfrak{p}_5$.

Example 9.2. We will factor $(5 + 2\sqrt{-14})$, which was left to the reader at the end of Example 4.30. Since $N(5 + 2\sqrt{-14}) = 81$, the only possible prime ideal factors of $(5 + 2\sqrt{-14})$ are \mathfrak{p}_3 and $\bar{\mathfrak{p}}_3$. It can't be divisible by both, since then it would be divisible by $\mathfrak{p}_3\bar{\mathfrak{p}}_3 = (3)$, and (3) doesn't divide $(5 + 2\sqrt{-14})$ because 3 doesn't divide $5 + 2\sqrt{-14}$ in $\mathbf{Z}[\sqrt{-14}]$. Therefore $(5 + 2\sqrt{-14})$ is a power of \mathfrak{p}_3 or a power of $\bar{\mathfrak{p}}_3$. Considering its norm is 81, $(5 + 2\sqrt{-14})$ is either \mathfrak{p}_3^4 or $\bar{\mathfrak{p}}_3^4$. Which is correct? Since $5 + 2\sqrt{-14} = 3 + 2(1 + \sqrt{-14}) \in \mathfrak{p}_3$, $\mathfrak{p}_3 \mid (5 + 2\sqrt{-14})$. Thus $(5 + 2\sqrt{-14}) = \mathfrak{p}_3^4$.

Example 9.3. The ideal $\mathfrak{a} = (2 + 3\sqrt{-14})$ has norm $130 = 2 \cdot 5 \cdot 13$. Therefore $\mathfrak{p}_2 \mid \mathfrak{a}$. Does \mathfrak{p}_5 or $\bar{\mathfrak{p}}_5$ divide \mathfrak{a} ? The ideal $(1 + \sqrt{-14})$ is divisible by \mathfrak{p}_5 and not by $\bar{\mathfrak{p}}_5$, so let's look at

the greatest common divisor of \mathfrak{a} and $(1 + \sqrt{-14})$, which is $(2 + 3\sqrt{-14}, 1 + \sqrt{-14})$. Its norm is 1, so $\bar{\mathfrak{p}}_5 \mid \mathfrak{a}$. To decide if \mathfrak{p}_{13} or $\bar{\mathfrak{p}}_{13}$ divides \mathfrak{a} , look at $(2 + 3\sqrt{-14}, 5 + \sqrt{-14})$, which is the greatest common divisor of \mathfrak{a} and $(5 + \sqrt{-14})$; it is \mathfrak{p}_{13} or (1) . By a calculation $(2 + 3\sqrt{-14}, 5 + \sqrt{-14})$ has norm 13, so $\mathfrak{p}_{13} \mid \mathfrak{a}$. Thus $\mathfrak{a} = \mathfrak{p}_2 \bar{\mathfrak{p}}_5 \mathfrak{p}_{13}$.

Example 9.4. The ideal $\mathfrak{a} = (7 + 3\sqrt{-14})$ has norm $175 = 5^2 \cdot 7$. Therefore $\mathfrak{p}_7 \mid \mathfrak{a}$. Either \mathfrak{p}_5 or $\bar{\mathfrak{p}}_5$ divides \mathfrak{a} (but not both, since otherwise their product (5) divides \mathfrak{a} , which is not the case). The ideal $(7 + 3\sqrt{-14}, 1 + \sqrt{-14})$ has norm 1, so \mathfrak{p}_5 does not divide \mathfrak{a} . Therefore $\mathfrak{a} = \bar{\mathfrak{p}}_5^2 \mathfrak{p}_7$.

As exercises, verify the prime ideal factorizations in Table 5. The first step is to compute the norm of each ideal to obtain a list of possible prime ideal factors from Table 4.

Ideal	Factorization
$(5 + \sqrt{-14})$	$\bar{\mathfrak{p}}_3 \mathfrak{p}_{13}$
$(2 + \sqrt{-14})$	$\mathfrak{p}_2 \bar{\mathfrak{p}}_3^2$
$(4 + \sqrt{-14})$	$\mathfrak{p}_2 \mathfrak{p}_3 \bar{\mathfrak{p}}_5$
$(7 + \sqrt{-14})$	$\mathfrak{p}_3^2 \mathfrak{p}_7$
$(7 + 2\sqrt{-14})$	$\bar{\mathfrak{p}}_3 \mathfrak{p}_5 \mathfrak{p}_7$
$(17 + 2\sqrt{-14})$	$\mathfrak{p}_3 \mathfrak{p}_5 \bar{\mathfrak{p}}_{23}$
$(20 + \sqrt{-14})$	$\mathfrak{p}_2 \bar{\mathfrak{p}}_3^2 \bar{\mathfrak{p}}_{23}$

TABLE 5. Factoring ideals in $\mathbf{Z}[\sqrt{-14}]$

These factorizations provide new ways of working out the ideal calculations in examples from Section 4. For instance, to redo Example 4.17 with Table 5, $(5 + \sqrt{-14}, 2 + \sqrt{-14}) = \bar{\mathfrak{p}}_3$ and $(4 + \sqrt{-14}, 2 - \sqrt{-14}) = \mathfrak{p}_2 \mathfrak{p}_3$ (note $\bar{\mathfrak{p}}_2 = \mathfrak{p}_2$), so the product of these ideals is $\mathfrak{p}_2 \mathfrak{p}_3 \bar{\mathfrak{p}}_3 = \mathfrak{p}_2(3) = (2, \sqrt{-14})(3) = (6, 3\sqrt{-14})$.

REFERENCES

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