SIMPLE RADICAL EXTENSIONS

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1. Introduction

A field extension L/K is called *simple radical* if $L = K(\alpha)$ where $\alpha^n = a$ for some $n \geq 1$ and $a \in K^\times$. Examples of simple radical extensions of $\mathbf{Q}$ are $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt[3]{2})$, and more generally $\mathbf{Q}(\sqrt[n]{2})$. A root of $T^n - a$ will be denoted $\sqrt[n]{a}$, so a simple radical extension of K looks like $K(\sqrt[n]{a})$, but the notation $\sqrt[n]{a}$ in general fields is ambiguous: different nth roots of a can generate different extensions of K, and they could even be nonisomorphic (*e.g.*, have different degrees over K) if $T^n - a$ is reducible in $K[T]$.

Example 1.1. In $\mathbf{C}$ the three roots of $T^3 - 8$ are 2, 2ω, and $2\omega^2$, where ω is a nontrivial cube root of unity; note $\omega^2 = 1/\omega$ and ω is a root of $(T^3 - 1)/(T - 1) = T^2 + T + 1$. While $\mathbf{Q}(2) = \mathbf{Q}$, the extension $\mathbf{Q}(2\omega) = \mathbf{Q}(\omega) = \mathbf{Q}(2/\omega)$ has degree 2 over $\mathbf{Q}$, so when the notation $\sqrt[3]{8}$ denotes some root of $T^3 - 8$ over $\mathbf{Q}$ then the field $\mathbf{Q}(\sqrt[3]{8})$ has two different meanings and $\mathbf{R}(\sqrt[3]{8})$ is $\mathbf{R}$ if $\sqrt[3]{8} = 2$ and it is $\mathbf{C}$ if $\sqrt[3]{8}$ is 2ω or $2\omega^2$.

Example 1.2. In the field $\mathbf{Q}(\sqrt{5})$ the number $2 + \sqrt{5}$ is a cube: $2 + \sqrt{5} = (\frac{1+\sqrt{5}}{2})^3$. The polynomial $T^3 - (2 + \sqrt{5})$ factors over $\mathbf{Q}(\sqrt{5})$ as

$$T^3 - (2 + \sqrt{5}) = \left(T - \frac{1 + \sqrt{5}}{2}\right)\left(T^2 + \frac{1 + \sqrt{5}}{2}T + \frac{3 + \sqrt{5}}{2}\right)$$

and the second factor is irreducible over $\mathbf{Q}(\sqrt{5})$ since it is irreducible over the larger field $\mathbf{R}$ (it is a quadratic with negative discriminant $-3(3+\sqrt{5})/2$). If $\sqrt[3]{2 + \sqrt{5}}$ means $(1+\sqrt{5})/2$ then $\mathbf{Q}(\sqrt[3]{2 + \sqrt{5}}) = \mathbf{Q}((1 + \sqrt{5})/2) = \mathbf{Q}(\sqrt{5})$, and if $\sqrt[3]{2 + \sqrt{5}}$ is a root of the quadratic factor of $T^3 - (2 + \sqrt{5})$ above then $\mathbf{Q}(\sqrt[3]{2 + \sqrt{5}})$ is a quadratic extension of $\mathbf{Q}(\sqrt{5})$.

We will focus here on the degree $[K(\sqrt[n]{a}) : K]$ and irreducibility relations for $T^n - a$ among different values of n, and intermediate fields between K and $K(\sqrt[n]{a})$.

2. Basic properties of $T^n - a$ and $\sqrt[n]{a}$

Theorem 2.1. *The degree $[K(\sqrt[n]{a}) : K]$ is at most n, and it equals n if and only if $T^n - a$ is irreducible over K, in which case the field $K(\sqrt[n]{a})$ up to isomorphism is independent of the choice of $\sqrt[n]{a}$.*

Proof. Since $\sqrt[n]{a}$ is a root of $T^n - a$, which is in $K[T]$, the minimal polynomial of $\sqrt[n]{a}$ over K is at most n, and thus $[K(\sqrt[n]{a}) : K] \leq n$.

If $[K(\sqrt[n]{a}) : K] = n$ then the minimal polynomial of $\sqrt[n]{a}$ over K has degree n, so it must be $T^n - a$ since that polynomial has degree n in $K[T]$ with $\sqrt[n]{a}$ as a root. As a minimal polynomial in $K[T]$ for some number, $T^n - a$ is irreducible over K.

Conversely, assume $T^n - a$ is irreducible over K. Then $\sqrt[n]{a}$ has minimal polynomial $T^n - a$ over K (the minimal polynomial of a number over K is the unique monic irreducible polynomial in $K[T]$ with that number as a root), so $[K(\sqrt[n]{a}) : K] = \deg(T^n - a) = n$.

When $T^n - a$ is irreducible over K, the field $K(\sqrt[n]{a})$ is isomorphic to $K[T]/(T^n - a)$ using evaluation at $\sqrt[n]{a}$ and thus, up to isomorphism (not up to equality!), the field $K(\sqrt[n]{a})$ is independent of the choice of $\sqrt[n]{a}$. □

Example 2.2. The polynomial $T^3 - 2$ is irreducible over $\mathbf{Q}$ and the three fields $\mathbf{Q}(\sqrt[3]{2})$, $\mathbf{Q}(\sqrt[3]{2}\omega)$, and $\mathbf{Q}(\sqrt[3]{2}\omega^2)$ are isomorphic to each other, where $\sqrt[3]{2}$ is the real cube root of 2 (or a cube root of 2 in an arbitrary field of characteristic 0) and ω is a nontrivial cube root of unity. This is no longer true if we replace $\mathbf{Q}$ by $\mathbf{R}$, since $T^3 - 2$ has one root in $\mathbf{R}$.

Theorem 2.3. *The roots of $T^n - a$ in a splitting field over K are numbers of the form $\zeta \sqrt[n]{a}$ where ζ is an nth root of unity ($\zeta^n = 1$) in K.*

Proof. Set $\alpha = \sqrt[n]{a}$, which is a fixed choice of root of $T^n - a$ over K. If β is another root in a splitting field of $T^n - a$ over K then $\beta^n = a = \alpha^n$, so $(\beta/\alpha)^n = 1$. Set $\zeta = \beta/\alpha \in K$, so $\beta = \zeta\alpha = \zeta \sqrt[n]{a}$ and $\zeta^n = (\beta/\alpha)^n = 1$.

Conversely, if $\zeta^n = 1$ and $\zeta \in K$ then $(\zeta \sqrt[n]{a})^n = \zeta^n a = a$, so $\zeta \sqrt[n]{a}$ is a root of $T^n - a$ in K. □

3. Prime exponents

In degree greater than 3, lack of roots ordinarily does *not* imply irreducibility. Consider $(T^2 - 2)(T^2 - 3)$ in $\mathbf{Q}[T]$. The polynomial $T^p - a$, where the exponent is prime, is a surprising counterexample: for these polynomials lack of a root is equivalent to irreducibility.

Theorem 3.1. *For an arbitrary field K and prime number p, and $a \in K^\times$, $T^p - a$ is irreducible in $K[T]$ if and only if it has no root in K. Equivalently, $T^p - a$ is reducible in $K[T]$ if and only if it has a root in K.*

Proof. Clearly if $T^p - a$ is irreducible in $K[T]$ then it has no root in K (since its degree is greater than 1).

In order to prove that $T^p - a$ not having a root in K implies it is irreducible we will prove the contrapositive: if $T^p - a$ is reducible in $K[T]$ then it has a root in K.

Write $T^p - a = g(T)h(T)$ in $K[T]$ where $m = \deg g$ satisfies $1 \leq m \leq p - 1$. Since $T^p - a$ is monic the leading coefficients of g and h multiply to 1, so by rescaling (which doesn't change degrees) we may assume g is monic and thus h is monic.

Let L be a splitting field of $T^p - a$ over K and $\alpha = \sqrt[p]{a}$ be one root of $T^p - a$ in L. Its other roots in L are $\zeta\alpha$ where $\zeta^p = 1$ (Theorem 2.3), so in $L[T]$

$$T^p - a = (T - \zeta_1\alpha)(T - \zeta_2\alpha) \cdots (T - \zeta_p\alpha)$$

where $\zeta_i^p = 1$. (Possibly $\zeta_i = \zeta_j$ when $i \neq j$; whether or not this happens doesn't matter.) By unique factorization in $L[T]$, every monic factor of $T^p - a$ in $L[T]$ is a product of some number of $(T - \zeta_i\alpha)$'s. Therefore

$$g(T) = (T - \zeta_{i_1}\alpha)(T - \zeta_{i_2}\alpha) \cdots (T - \zeta_{i_m}\alpha) \tag{3.1}$$

for some pth roots of unity $\zeta_{i_1}, \ldots, \zeta_{i_m}$.

Now let's *look at the constant terms* in (3.1). Set $c = g(0)$, so

$$c = (-1)^m(\zeta_{i_1} \cdots \zeta_{i_m})\alpha^m.$$

Since $g(T) \in K[T]$, $c \in K$ and $c \neq 0$ on account of $g(0)h(0) = 0^p - a = -a$. Therefore

$$c = (-1)^m(\zeta_{i_1} \cdots \zeta_{i_m})\alpha^m \in K^\times. \tag{3.2}$$

We want to replace α^m with α, and will do this by raising α^m to an additional power to make the exponent on α congruent to 1 mod p.

Because p is prime and $1 \leq m \leq p-1$, m and p are relatively prime: we can write $mx + py = 1$ for some x and y in $\mathbf{Z}$. Raise the product in (3.2) to the x-power to make the exponent on α equal to $mx = 1 - py$:

$$\begin{aligned} c^x &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \alpha^{mx} \\ &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \alpha^{1-py} \\ &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \frac{\alpha}{(\alpha^p)^y} \\ &= (-1)^{mx}(\zeta_{i_1} \cdots \zeta_{i_m})^x \frac{\alpha}{a^y}, \end{aligned}$$

so

$$(\zeta_{i_1} \cdots \zeta_{i_m})^x \alpha = a^y(-1)^{mx} c^x \in K^\times$$

and the left side has the form $\zeta\alpha$ where $\zeta^p = 1$, so K contains a root of $T^p - a$. □

Remark 3.2. For an odd prime p and a field K, the irreducibility of $T^p - a$ over K implies irreducibility of $T^{p^r} - a$ for all $r \geq 1$, which is not obvious! And this doesn't quite work when $p = 2$: irreducibility of $T^4 - a$ implies irreducibility of $T^{2^r} - a$ for all $r \geq 2$ (again, not obvious!), but irreducibility of $T^2 - a$ need not imply irreducibility of $T^4 - a$. A basic example is that $T^2 + 4$ is irreducible in $\mathbf{Q}[T]$ but $T^4 + 4 = (T^2 + 2T + 2)(T^2 - 2T + 2)$. See [2, pp. 297–298] for a precise irreducibility criterion for $T^n - a$ over a general field, which is due to Vahlen [4] in 1895 for $K = \mathbf{Q}$, Capelli [1] in 1897 for K of characteristic 0, and Rédei [3] in 1959 for positive characteristic.

4. Irreducibility relations among $T^n - a$ for different exponents

Theorem 4.1. *Let K be a field, $a \in K^\times$, and assume $T^n - a$ is irreducible over K. If $d \mid n$ then $T^d - a$ is irreducible over K. Equivalently, if $[K(\sqrt[n]{a}) : K] = n$ for some nth root of a over K then for all $d \mid n$ we have $[K(\sqrt[d]{a}) : K] = d$ for every dth root of a.*

Proof. We prove irreducibility of $T^n - a$ implies irreducibility of $T^d - a$ in two ways: working with polynomials and working with field extensions.

Polynomials: assume $T^d - a$ is reducible over K, so $T^d - a = g(T)h(T)$ where $0 < \deg g(T) < d$. Replacing T with $T^{n/d}$ in this equation, we get $T^n - a = g(T^{n/d})h(T^{n/d})$ where $\deg g(T^{n/d}) = (n/d)\deg g < (n/d)d = n$ and clearly $\deg g(T^{n/d}) > 0$.

Field extensions: let $\sqrt[n]{a}$ be an nth root of a over K, so $[K(\sqrt[n]{a}) : K] = n$ by Theorem 2.1. Define $\sqrt[d]{a} = \sqrt[n]{a}^{\,n/d}$. This is a root of $T^d - a$ since $\sqrt[d]{a}^{\,d} = (\sqrt[n]{a}^{\,n/d})^d = \sqrt[n]{a}^{\,n} = a$. To prove $T^d - a$ is irreducible over K we will prove $[K(\sqrt[d]{a}) : K] = d$ using that choice of $\sqrt[d]{a}$.

In the tower $K \subset K(\sqrt[d]{a}) \subset K(\sqrt[n]{a})$, we have $[K(\sqrt[d]{a}) : K] \leq d$ and $[K(\sqrt[n]{a}) : K(\sqrt[d]{a})] \leq n/d$ by Theorem 2.1, since $\sqrt[d]{a}$ is a root of $T^d - a \in K[T]$ and $\sqrt[n]{a}$ is a root of $T^{n/d} - \sqrt[d]{a} \in K(\sqrt[d]{a})[T]$. We have

$$[K(\sqrt[n]{a}) : K] = [K(\sqrt[n]{a}) : K(\sqrt[d]{a})][K(\sqrt[d]{a}) : K]$$

and our irreducibility hypothesis implies the left side is n, so it follows that our upper bounds n/d and d for the factors on the right must be equalities. In particular, $[K(\sqrt[d]{a}):K] = d$ so $T^d - a$ is irreducible over K (it has a root with degree d over K). $\square$

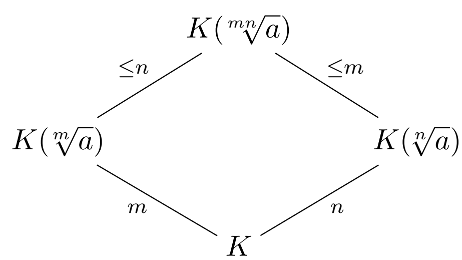
There was an important calculation in this proof that we will use repeatedly below: if $d \mid n$ then $K(\sqrt[n]{a})$ contains $K(\sqrt[d]{a})$, where $\sqrt[d]{a} := \sqrt[n]{a}^{n/d}$. This is a root of $T^d - a$, so the notation is reasonable, but note that $\sqrt[d]{a}$ is not an arbitrary dth root of a: it depends on the choice made first of $\sqrt[n]{a}$.

By Theorem 4.1 and Remark 3.2, for odd primes p irreducibility of $T^p - a$ is equivalent to irreducibility of $T^{p^r} - a$ for a single $r \geq 1$, and for the prime 2 irreducibility of $T^4 - a$ is equivalent to irreducibility of $T^{2^r} - a$ for a single $r \geq 2$.

Theorem 4.2. *For relatively prime positive integers m and n, $T^{mn} - a$ is irreducible over K if and only if $T^m - a$ and $T^n - a$ are each irreducible over K. Equivalently, if m and n are relatively prime positive integers then $[K(\sqrt[mn]{a}):K] = mn$ if and only if $[K(\sqrt[m]{a}):K] = m$ and $[K(\sqrt[n]{a}):K] = n$.*

Proof. That irreducibility of $T^{mn} - a$ over K implies irreducibility of $T^m - a$ and $T^n - a$ over K follows from Theorem 4.1.

To prove irreducibility of $T^m - a$ and $T^n - a$ over K implies irreducibility of $T^{mn} - a$ over K we will work with roots of these polynomials. It is convenient to select mth, nth, and mnth roots of a in a multiplicatively compatible way: fix a root $\sqrt[mn]{a}$ of $T^{mn} - a$ over K and define $\sqrt[m]{a} := \sqrt[mn]{a}^{\,n}$ and $\sqrt[n]{a} := \sqrt[mn]{a}^{\,m}$. Then $\sqrt[m]{a}$ is a root of $T^m - a$ and $\sqrt[n]{a}$ is a root of $T^n - a$, so we have the following field diagram, where the containments are due to $\sqrt[m]{a}$ and $\sqrt[n]{a}$ being powers of $\sqrt[mn]{a}$.



The bottom field degree values come from $T^m - a$ and $T^n - a$ being irreducible over K, and the top field degree upper bounds come from $\sqrt[mn]{a}$ being a root of $T^n - \sqrt[m]{a} \in K(\sqrt[m]{a})[T]$ and $T^m - \sqrt[n]{a} \in K(\sqrt[n]{a})[T]$. Let $d = [K(\sqrt[mn]{a}):K]$, so by reading the field diagram along either the left or right we have $\boxed{d \leq mn}$. Also d is divisible by m and by n since field degrees are multiplicative in towers, so from relative primality of m and n we get $m \mid d, n \mid d \Longrightarrow mn \mid d$, so $\boxed{mn \leq d}$. Thus $d = mn$, so $T^{mn} - a$ is the minimal polynomial of $\sqrt[mn]{a}$ over K and thus is irreducible over K. $\square$

Corollary 4.3. *For an integer $N > 1$ with prime factorization $p_1^{e_1} \cdots p_k^{e_k}$, $T^N - a$ is irreducible over K if and only if each $T^{p_i^{e_i}} - a$ is irreducible over K.*

Proof. Use Theorem 4.2 with the factorization $N = p_1^{e_1}(p_2^{e_2} \cdots p_k^{e_k})$ to see irreducibility of $T^N - a$ over K is equivalent to irreducibility of $T^{p_1^{e_1}} - a$ and $T^{p_2^{e_2} \cdots p_k^{e_k}} - a$ over K, and then by induction on the number of different prime powers in the degree, irreducibility of $T^{p_2^{e_2} \cdots p_k^{e_k}} - a$ over K is equivalent to irreducibility of $T^{p_i^{e_i}} - a$ over K for $i = 2, \ldots, k$. $\square$

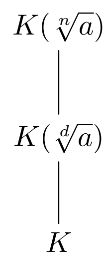
Example 4.4. Irreducibility of $T^{90} - a$ over K is equivalent to irreducibility of $T^2 - a$, $T^9 - a$, and $T^5 - a$ over K.

Remark 4.5. By Remark 3.2, if N is odd then irreducibility of $T^N - a$ over K is equivalent to irreducibility of $T^{p_i} - a$ over K as p_i runs over the prime factors of N (the multiplicities e_i don't matter!), and for these we know the story for irreducibility by Theorem 3.1: it's the same thing as $T^{p_i} - a$ not having a root in K for each p_i.

Example 4.6. Irreducibility of $T^{75} - a$ over K is equivalent to a not having a cube root or fifth root in K.

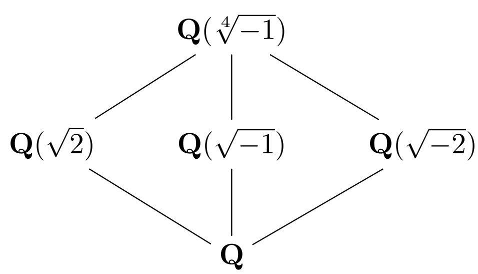
5. Intermediate fields in a simple radical extension

For a choice of nth root $\sqrt[n]{a}$ and a factor $d \mid n$, $\sqrt[d]{a} := \sqrt[n]{a}^{n/d}$ is a root of $T^d - a$ in $K(\sqrt[n]{a})$, so we have the following field diagram.



It's natural to ask if every field between K and $K(\sqrt[n]{a})$ is $K(\sqrt[d]{a})$ for some d dividing n. The simplest setting to study this is when $T^n - a$ is irreducible over K (and thus also $T^d - a$ is irreducible over K, by Theorem 4.1), so $[K(\sqrt[d]{a}) : K] = d$. Is $K(\sqrt[d]{a})$ the only extension of K of degree d inside $K(\sqrt[n]{a})$? This is not always true.

Example 5.1. Let $K = \mathbf{Q}$ and consider the field $\mathbf{Q}(\sqrt[4]{-1})$. Set $\alpha = \sqrt[4]{-1}$, so $\alpha^4 + 1 = 0$. The polynomial $T^4 + 1$ is irreducible over $\mathbf{Q}$ because it becomes Eisenstein at 2 when T is replaced with $T + 1$. Since $[\mathbf{Q}(\sqrt[4]{-1}) : \mathbf{Q}] = 4$, the fields strictly between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[4]{-1})$ are quadratic over $\mathbf{Q}$. One of these is $\mathbf{Q}(\sqrt{-1})$, but it is not the only one.

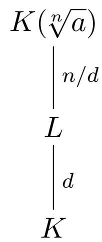


If $\alpha^4 = -1$ then $(\alpha + 1/\alpha)^2 = \alpha^2 + 2 + 1/\alpha^2 = (\alpha^4 + 1)/\alpha^2 + 2 = 2$ and $(\alpha - 1/\alpha)^2 = \alpha^2 - 2 + 1/\alpha^2 = (\alpha^4 + 1)/\alpha^2 - 2 = -2$, so $\mathbf{Q}(\sqrt[4]{-1})$ contains $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{-2})$. None of the fields $\mathbf{Q}(i)$, $\mathbf{Q}(\sqrt{2})$, and $\mathbf{Q}(\sqrt{-2})$ are the same, so we have at least three (and in fact there are just these three) quadratic extensions of $\mathbf{Q}$ in $\mathbf{Q}(\sqrt[4]{-1})$.

In the above example, the "reason" for the appearance of more intermediate fields between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[4]{-1})$ than just $\mathbf{Q}(\sqrt{-1})$ is that there are 4th roots of unity in $\mathbf{Q}(\sqrt[4]{-1})$ that are not in $\mathbf{Q}$, namely $\pm\sqrt{-1}$. The following theorem shows we get no such unexpected fields if all nth roots of unity in the top field are actually in the base field.

Theorem 5.2. *Let K be a field, $a \in K^\times$, and assume $T^n - a$ is irreducible over K. If all nth roots of unity in $K(\sqrt[n]{a})$ are in K then for each $d \mid n$ the only field between K and $K(\sqrt[n]{a})$ of degree d over K is $K(\sqrt[d]{a})$, where $\sqrt[d]{a} := \sqrt[n]{a}^{n/d}$.*

Proof. Every field between K and $K(\sqrt[n]{a})$ has degree over K that divides n. For $d \mid n$ suppose L is a field with $K \subset L \subset K(\sqrt[n]{a})$ and $[L : K] = d$. To prove $L = K(\sqrt[d]{a})$, it suffices to show $\sqrt[d]{a} \in L$, since that would give us $K(\sqrt[d]{a}) \subset L$ and we know $K(\sqrt[d]{a})$ has degree d over K, so the containment $K(\sqrt[d]{a}) \subset L$ would have to be an equality.



Let $f(T)$ be the minimal polynomial of $\sqrt[n]{a}$ **over** L, so $f(T) \mid (T^n - a)$ and $\deg f = n/d$. We can write another root of $f(T)$ as $\zeta \sqrt[n]{a}$ for some nth root of unity ζ. (Theorem 2.3). In a splitting field of $T^n - a$ over K, the factorization of $f(T)$ is $\prod_{i \in I}(T - \zeta_i \sqrt[n]{a})$ for some nth roots of unity ζ_i (I is just an index set). The constant term of $f(T)$ is in L, so $(\prod_{i \in I} \zeta_i) \sqrt[n]{a}^{n/d} \in L$. Therefore $(\prod_{i \in I} \zeta_i) \sqrt[n]{a}^{n/d} \in K(\sqrt[n]{a})$, so $\prod_{i \in I} \zeta_i \in K(\sqrt[n]{a})$. The only nth roots of unity in $K(\sqrt[n]{a})$ are, by hypothesis, in K, so $\prod_{i \in I} \zeta_i \in K \subset L$. Therefore $\sqrt[n]{a}^{n/d} = \sqrt[d]{a}$ is in L, so we're done. $\square$

Example 5.3. If $K = \mathbf{Q}$, $a > 0$, and $T^n - a$ is irreducible over $\mathbf{Q}$ then $\mathbf{Q}(\sqrt[n]{a})$ is isomorphic to a subfield of $\mathbf{R}$ (using the *real* positive nth root of a), which implies the only roots of unity in $\mathbf{Q}(\sqrt[n]{a})$ are ± 1 and those both lie $\mathbf{Q}$. For example, the only fields between $\mathbf{Q}$ and $\mathbf{Q}(\sqrt[n]{2})$ are $\mathbf{Q}(\sqrt[d]{2})$ where $d \mid n$ and $\sqrt[d]{2} = \sqrt[n]{2}^{n/d}$.

Example 5.4. Let F be a field and $K = F(u)$, the rational functions over F in one indeterminate. The polynomial $T^n - u$ is irreducible over $F(u)$ since it is Eisenstein at u. We let $\sqrt[n]{u}$ denote one root of $T^n - u$, so $K(\sqrt[n]{u}) = F(\sqrt[n]{u})$ has degree n over $F(u)$. All roots of unity in $F(\sqrt[n]{u})$ – not just nth roots of unity – are in F, because $F(\sqrt[n]{u})$ is itself a rational function field in one indeterminate over F (since $\sqrt[n]{u}$ is transcendental over F) and all elements of a rational function field in one indeterminate over F that are not in F are transcendental over F and thus can't be a root of unity. Therefore by Theorem 5.2, the fields between $F(u)$ and $F(\sqrt[n]{u})$ are $F(\sqrt[d]{u})$ for $d \mid n$.

Example 5.5. An example where the hypothesis that all nth roots of unity in $K(\sqrt[n]{a})$ are in K is false, yet the conclusion of Theorem 5.2 is true, is $K = \mathbf{Q}(i)$, $a = 2$, and $n = 8$: it can be shown that $[\mathbf{Q}(i, \sqrt[8]{2}) : \mathbf{Q}(i)] = 8$ and the only fields between $\mathbf{Q}(i)$ and $\mathbf{Q}(i, \sqrt[8]{2})$ are $\mathbf{Q}(i, \sqrt[d]{2})$ for $d = 1, 2, 4, 8$ while $\frac{1+i}{\sqrt{2}}$ is an 8th root of unity in $\mathbf{Q}(\sqrt[8]{2}, i)$ that is not in $\mathbf{Q}(i)$.

References

[1] A. Capelli, *Sulla riduttibilità delle equazioni algebriche, Nota prima*, Rend. Accad. Sci. Fis. Mat. Soc. Napoli **3** (1897), 243–252.
[2] S. Lang, "Algebra," 3rd revised ed., Springer-Verlag, New York, 2002.
[3] L. Rédei, "Algebra," Erster Teil, Akad. Verlagsges, Geest & Portig K.-G., Leipzig, 1959.
[4] K. T. Vahlen, *Über reductible Binome*, Acta Math. **19** (1895), 195–198.