

ESTIMATING DEFINITE INTEGRALS OF e^{-t^2} IN TWO WAYS

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The function e^{-t^2} is important since its graph defines the bell curve in probability theory¹, which relates the area under its graph to the normal distribution. However, e^{-t^2} has no concrete antiderivative, so $\int_a^b e^{-t^2} dt$ when $a < b$ is found by approximations.

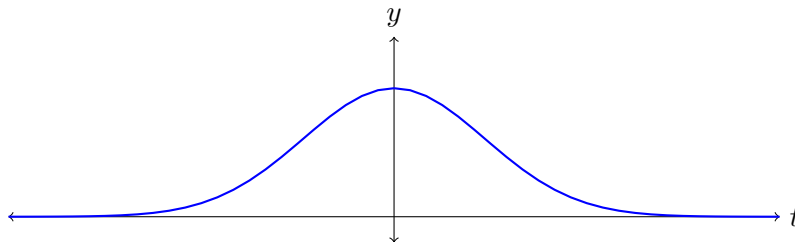


FIGURE 1. Plot of $y = e^{-t^2}$.

We will approximate $\int_0^T e^{-t^2} dt$ when $T > 0$ in two ways, power series and integration by parts, and apply each to estimate $\int_0^3 e^{-t^2} dt$. The methods are taken from a 19th century astronomy book [1, pp. 154-155], where $\int_0^T e^{-t^2} dt$ is used in calculations related not to probability, but to light refraction (how light rays pass through our atmosphere). We will bound errors in these approximations as in [1], where the author writes in the preface

When approximate methods are employed ... their degree of accuracy is carefully determined by ... converging series, and only such terms [are] neglected as can be shown to be insensitive in the cases to which the formulas are to be applied.

Method 1: Series.

Since $e^x = \sum_{n \geq 0} \frac{x^n}{n!}$ for all x , $e^{-t^2} = \sum_{n \geq 0} \frac{(-t^2)^n}{n!} = \sum_{n \geq 0} \frac{(-1)^n}{n!} t^{2n}$. By termwise integration,

$$\begin{aligned}
 \int_0^T e^{-t^2} dt &= \int_0^T \left(\sum_{n \geq 0} \frac{(-1)^n}{n!} t^{2n} \right) dt \\
 &= \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_0^T t^{2n} dt \\
 &= \sum_{n \geq 0} \frac{(-1)^n T^{2n+1}}{n!(2n+1)} \\
 &= T - \frac{T^3}{3} + \frac{T^5}{2 \cdot 5} - \frac{T^7}{6 \cdot 7} + \frac{T^9}{24 \cdot 9} - \frac{T^{11}}{120 \cdot 11} + \cdots
 \end{aligned}
 \tag{1}$$

¹The standard bell curve is actually the graph of $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ rather than e^{-t^2} , but we'll work with e^{-t^2} to avoid extra constant factors in our calculations.

When $T > 0$ this is an alternating series whose successive terms decrease in absolute value, so

$$\int_0^T e^{-t^2} dt = \sum_{n=0}^N \frac{(-1)^n T^{2n+1}}{n!(2n+1)} + R_N,$$

where

$$|R_N| \leq \left| \frac{(-1)^{N+1} T^{2(N+1)+1}}{(N+1)!(2(N+1)+1)} \right| = \frac{T^{2N+3}}{(N+1)!(2N+3)}.$$

Take $T = 3$, so $\int_0^3 e^{-t^2} dt = s_N + R_N$, where $s_N = \sum_{n=0}^N \frac{(-1)^n 3^{2n+1}}{n!(2n+1)}$ and $|R_N| \leq \frac{3^{2N+3}}{(N+1)!(2N+3)}$. The table below presents these values when $N = 10, 20, 30$, and 40 .

N	s_N	$ R_N $ bound
10	60.7	1127.9
20	.993	3.1377
30	.88620908	6.8491
40	.886207348260	$5.89 \cdot 10^{-11}$

Using the error bound when $N = 40$,

$$(2) \quad .88620734820 < \int_0^3 e^{-t^2} dt < .88620734832,$$

which gives us the first 9 decimal places in the integral: .886207348.

Method 2: Integration by parts.

Apply integration by parts to $\int_0^T e^{-t^2} dt$ with $u = e^{-t^2}$ and $dv = dt$, so $du = -2te^{-t^2} dt$ and $v = t$:

$$(3) \quad \int_0^T e^{-t^2} dt = uv \Big|_{t=0}^{t=T} - \int_0^T v du = Te^{-T^2} + 2 \int_0^T t^2 e^{-t^2} dt.$$

Use integration by parts again in the new integral, with $u = e^{-t^2}$ as before and $dv = t^2 dt$, so $du = -2te^{-t^2} dt$ and $v = t^3/3$:

$$\int_0^T t^2 e^{-t^2} dt = uv \Big|_{t=0}^{t=T} - \int_0^T v du = \frac{T^3}{3} e^{-T^2} + \frac{2}{3} \int_0^T t^4 e^{-t^2} dt.$$

Substituting this into (3),

$$(4) \quad \begin{aligned} \int_0^T e^{-t^2} dt &= Te^{-T^2} + 2 \left(\frac{T^3}{3} e^{-T^2} + \frac{2}{3} \int_0^T t^4 e^{-t^2} dt \right) \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3} \int_0^T t^4 e^{-t^2} dt. \end{aligned}$$

We've faced $\int_0^T e^{-t^2} dt$, $\int_0^T t^2 e^{-t^2} dt$, and now $\int_0^T t^4 e^{-t^2} dt$. Let's just consider the general integral

$$I_n = \int_0^T t^{2n} e^{-t^2} dt$$

with an even exponent $2n$, where $n \geq 0$. Use integration by parts on this with $u = e^{-t^2}$ and $dv = t^{2n} dt$, so $du = -2te^{-t^2} dt$ and $v = t^{2n+1}/(2n+1)$:

$$I_n = \int_0^T t^{2n} e^{-t^2} dt = uv \Big|_{t=0}^{t=T} - \int_0^T v du = \frac{T^{2n+1}}{2n+1} e^{-T^2} + \frac{2}{2n+1} \int_0^T t^{2(n+1)} e^{-t^2} dt,$$

so

$$(5) \quad I_n = \frac{T^{2n+1}}{2n+1} e^{-T^2} + \frac{2}{2n+1} I_{n+1}.$$

We'll use this recursion to recover (3) and (4) and go further:

$$\begin{aligned} \int_0^T e^{-t^2} dt &= I_0 \\ &= Te^{-T^2} + 2I_1 \\ &= Te^{-T^2} + 2 \left(\frac{T^3}{3} e^{-T^2} + \frac{2}{3} I_2 \right) \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3} I_2 \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3} \left(\frac{T^5}{5} e^{-T^2} + \frac{2}{5} I_3 \right) \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3 \cdot 5} T^5 e^{-T^2} + \frac{2^3}{3 \cdot 5} I_3 \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3 \cdot 5} T^5 e^{-T^2} + \frac{2^3}{3 \cdot 5} \left(\frac{T^7}{7} e^{-T^2} + \frac{2}{7} I_4 \right) \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3 \cdot 5} T^5 e^{-T^2} + \frac{2^3}{3 \cdot 5 \cdot 7} T^7 e^{-T^2} + \frac{2^4}{3 \cdot 5 \cdot 7} I_4 \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3 \cdot 5} T^5 e^{-T^2} + \frac{2^3}{3 \cdot 5 \cdot 7} T^7 e^{-T^2} + \frac{2^4}{3 \cdot 5 \cdot 7} \left(\frac{T^9}{9} e^{-T^2} + \frac{2}{9} I_5 \right) \\ &= Te^{-T^2} + \frac{2}{3} T^3 e^{-T^2} + \frac{2^2}{3 \cdot 5} T^5 e^{-T^2} + \frac{2^3}{3 \cdot 5 \cdot 7} T^7 e^{-T^2} + \frac{2^4}{3 \cdot 5 \cdot 7 \cdot 9} T^9 e^{-T^2} + \frac{2^5}{3 \cdot 5 \cdot 7 \cdot 9} I_5. \end{aligned}$$

The general pattern here is

$$(6) \quad \boxed{\int_0^T e^{-t^2} dt = e^{-T^2} \sum_{n=0}^N \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} T^{2n+1} + \tilde{R}_N,}$$

where the sum over $0 \leq n \leq N$ has all positive terms (it is not an alternating series) and

$$\tilde{R}_N := \frac{2^{N+1}}{1 \cdot 3 \cdot 5 \cdots (2N+1)} I_{N+1} = \frac{2^{N+1}}{1 \cdot 3 \cdot 5 \cdots (2N+1)} \int_0^T t^{2(N+1)} e^{-t^2} dt.$$

The remainder $\tilde{R}_N$ is positive. To bound it from above, let's bound the integrand $t^{2(N+1)} e^{-t^2}$ on $[0, T]$. Call that integrand $f_N(t)$. It vanishes at $t = 0$, is positive when $t > 0$, and a calculation left to the reader shows

$$f'_N(t) = 2t^{2N+1} e^{-t^2} (N+1 - t^2),$$

so $f'_N(t) > 0$ when $0 < t < \sqrt{N+1}$, $f'_N(\sqrt{N+1}) = 0$, and $f'_N(t) < 0$ when $t > \sqrt{N+1}$. Thus f_N is increasing when $0 < t < \sqrt{N+1}$ and is maximized at $t = \sqrt{N+1}$, so when $T \leq \sqrt{N+1}$ (equivalently, $N+1 \geq T^2$) and $0 \leq t \leq T$, we have

$$0 \leq f_N(t) \leq f_N(T) = T^{2(N+1)} e^{-T^2}.$$

Therefore

$$N + 1 \geq T^2 \implies \int_0^T t^{2(N+1)} e^{-t^2} dt \leq T^{2(N+1)} e^{-T^2} \int_0^T dt = T^{2N+3} e^{-T^2},$$

so

$$\begin{aligned} \tilde{R}_N &= \frac{2^{N+1}}{1 \cdot 3 \cdot 5 \cdots (2N+1)} \int_0^T t^{2(N+1)} e^{-t^2} dt \\ &\leq \frac{2^{N+1}}{3 \cdot 5 \cdots (2N+1)} T^{2N+3} e^{-T^2} \\ &= T e^{-T^2} \frac{(2T^2)^{N+1}}{1 \cdot 3 \cdot 5 \cdots (2N+1)} \end{aligned}$$

when $N + 1 \geq T^2$. This bound on $\tilde{R}_N$ (with fixed T) tends to 0 as $N \rightarrow \infty$, so letting $N \rightarrow \infty$ in (6) gives us a second formula for $\int_0^T e^{-t^2} dt$ using an infinite series of positive terms with a factor e^{-T^2} out front:

$$\begin{aligned} \int_0^T e^{-t^2} dt &= e^{-T^2} \sum_{n \geq 0} \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} T^{2n+1} \\ &= e^{-T^2} \left(T + \frac{2}{3} T^3 + \frac{2^2}{3 \cdot 5} T^5 + \frac{2^3}{3 \cdot 5 \cdot 7} T^7 + \frac{2^4}{3 \cdot 5 \cdot 7 \cdot 9} T^9 + \frac{2^5}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} T^{11} + \cdots \right). \end{aligned}$$

Compare this to (1), which is an alternating series.

In (6) set $T = 3$, so $\int_0^3 e^{-t^2} dt = \tilde{s}_N + \tilde{R}_N$, where $\tilde{s}_N = e^{-9} \sum_{n=0}^N \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} 3^{2n+1}$

and $0 < \tilde{R}_N \leq 3e^{-9} \frac{18^{N+1}}{1 \cdot 3 \cdot 5 \cdots (2N+1)}$. The table below has these values when $N = 10, 20, 30$, and 40 (so $N + 1 \geq 3^2 = 9$).

N	$\tilde{s}_N$	$\tilde{R}_N$ bound
10	.6712	1.7305
20	.885960	.0064787
30	.8862073445	$1.702 \cdot 10^{-7}$
40	.886207348259518	$1.676 \cdot 10^{-13}$

Using the error bound when $N = 40$,

$$.886207348259350 < \int_0^3 e^{-t^2} dt < .886207348259685,$$

which gives us the first 12 decimal places in the integral: .886207348259. Our previous estimate, with $N = 40$ in (2), gave us the first 9 decimal places in this integral.

REFERENCES

- [1] W. Chauvenet, *A Manual of Spherical and Practical Astronomy*, Volume 1, J. B. Lippincott & Co., Philadelphia, 1863. URL <https://books.google.com/books?id=JeokvgAACAAJ>.