

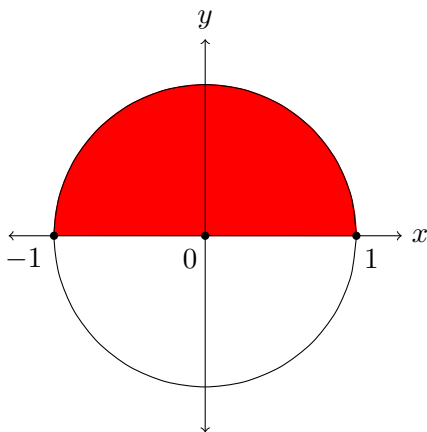
ARC LENGTH, INTEGRATION BY PARTS, AND π

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The number π appears in formulas for two properties of a circle: its area and its circumference. A circle of radius r has area πr^2 and circumference $2\pi r$. Why does the same number π show up in both formulas? It's not a surprise that π occurs in the circumference formula, since that is more or less the way π is defined when we first learn about it: the ratio of the circumference C to the diameter $D = 2r$ for any circle is defined to be π , and the equation $\pi = C/D$ is the same as $C = \pi D = \pi(2r) = 2\pi r$.

To explain why π shows up in the formula for the area of a circle as well, we will focus on a special case of circles with radius 1 (to keep the notation simpler), for which the circumference is 2π by definition and we want to show the area is π . We will express the area A and circumference C of this circle as integrals, using the arc length integral in the case of C , and then use *integration by parts* to show $C = 2A$, so from $C = 2\pi$ we get $A = \pi$.

To express C and A as integrals, we will work with half a circle, as shaded below. The circle has area twice the shaded region and circumference equal to twice the arc length from $(1,0)$ to $(-1,0)$. Both of these can be expressed as integrals.



The area of the shaded region is the area under the circular arc from $x = -1$ to $x = 1$. The equation of the circular arc along the top of the shaded region is $y = \sqrt{1 - x^2}$, so

$$(1) \quad A = 2 \int_{-1}^1 \sqrt{1 - x^2} \, dx.$$

The length of the circular arc along the top of the shaded region is the length of the curve $y = \sqrt{1 - x^2}$ from $x = -1$ to $x = 1$, which is $\int_{-1}^1 \sqrt{1 + (dy/dx)^2} \, dx$. When $y = \sqrt{1 - x^2}$, we have

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}} \implies \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1 - x^2} \implies 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2},$$

so

$$(2) \quad C = 2 \int_{-1}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2 \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}}.$$

Using (1) and (2),

$$C = 2A \iff 2 \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = 4 \int_{-1}^1 \sqrt{1-x^2} dx \iff \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = 2 \int_{-1}^1 \sqrt{1-x^2} dx.$$

To prove $C = 2A$, we will prove that last equation:

$$(3) \quad \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \stackrel{?}{=} 2 \int_{-1}^1 \sqrt{1-x^2} dx.$$

In the integral on the right side of (3) use integration by parts with

$$u = \sqrt{1-x^2} \quad \text{and} \quad dv = dx.$$

Then

$$du = \frac{-2x}{2\sqrt{1-x^2}} dx = -\frac{x}{\sqrt{1-x^2}} dx \quad \text{and} \quad v = x,$$

so

$$(4) \quad \int_{-1}^1 \sqrt{1-x^2} dx = \int_{-1}^1 u dv = uv \Big|_{-1}^1 - \int_{-1}^1 v du = 0 + \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx.$$

Simplify the last integrand by writing the numerator x^2 as $x^2 - 1 + 1$:

$$\begin{aligned} \int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx &= \int_{-1}^1 \frac{x^2 - 1 + 1}{\sqrt{1-x^2}} dx \\ &= \int_{-1}^1 \frac{x^2 - 1}{\sqrt{1-x^2}} dx + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx \\ &= \int_{-1}^1 -\frac{(1-x^2)}{\sqrt{1-x^2}} dx + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx \\ &= -\int_{-1}^1 \sqrt{1-x^2} dx + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx, \end{aligned}$$

and feeding this into (4) gives us

$$\int_{-1}^1 \sqrt{1-x^2} dx = -\int_{-1}^1 \sqrt{1-x^2} dx + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx.$$

Adding $\int_{-1}^1 \sqrt{1-x^2} dx$ to both sides gives us (3), which completes the proof that if we define π so that a circle with radius 1 has circumference 2π then the area of a circle with radius 1 is π .